COHEN-MACAULAYNESS OF MODULES OF COVARIANTS

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1. Introduction

Let G be a connected reductive algebraic group over an algebraically closed field k of characteristic zero and let W be a finite dimensional representation of G. Then G acts on the polynomial ring $R = SW$ and the Hochster-Roberts theorem asserts that R^G is Cohen-Macaulay.

A natural question would be whether the same is true for modules of covariants, i.e. let U be another finite dimensional G-representation. Then $U \otimes_k R$ is a free R-module and a natural generalization of the Hochster-Roberts theorem would be that $(U \otimes_k R)^G$ is a Cohen-Macaulay R^G-module. Unfortunately, it is easy to see that this cannot be true in general (see [10, Ex. 3.1] for a simple counter example).

For an arbitrary irreducible character χ, Stanley defines R^G_χ as the sum of all irreducible subrepresentations of R, with character χ [7].

Assume that U is irreducible and let χ be the character of U^*. Then it is easy to see that

$$R^G_\chi \cong U^* \otimes (U \otimes R)^G$$

as R^G-modules. Hence the Cohen-Macaulayness of R^G_χ is equivalent with that of $(U \otimes R)^G$.

There is a well known conjecture, due to Stanley, that gives sufficient conditions for R^G_χ to be Cohen-Macaulay [7]. This conjecture was proved by him if G is a torus.

Let $T \subset G$ be a maximal torus and let $X(T)$ be the character group of T. The group law on $X(T)$ will be written additively. Let $\alpha_1, \dots, \alpha_d \in X(T)$ be the weights of W (taken with multiplicity).

Let $\chi \in X(T)$. Then Stanley says that χ is *critical for* (T, W) if the following condition is satisfied :

The system

$$z_1\alpha_1 + \cdots + z_d\alpha_d = \chi \tag{1}$$

has a rational (or equivalently real) solution $(u_1, \dots, u_d)$ with the following properties :

- $u_i \leq 0$
- if $(b_1, \dots, b_d)$ is an integer solution to (1) such that $b_i \geq u_i$ then $b_i \geq 0$ for all i.

(Note that it makes sense to speak of a rational solution to (1) because $X(T)$ is a torsion free abelian group.)

A character is clearly critical if it is of the form $\sum u_i\alpha_i$, $-1 < u_i \leq 0$. We will call such a character *strongly critical for* (T, W). This notion is useful because it is somewhat easier to check that a character is strongly critical than that it is critical.

Let χ be an irreducible character of G. Then $\chi \mid T = \chi_1 + \cdots + \chi_u$ where $\chi_i \in X(T)$. Let Φ be the set of roots of G. Then Stanley calls χ *critical for* (G, W) if $\chi_i - \sum_{\rho\in S}\rho$ is critical for (T, W) for all $1 \leq i \leq u$ and for all $S \subset \Phi$.

Again we will say that χ is *strongly critical for* (G, W) if all $\chi_i - \sum_{\rho\in S}\rho$ are strongly critical for (T, W).

Conjecture 1.1. [7] If χ is critical for (G, W) then R^G_χ is Cohen-Macaulay.

In this paper we will prove a different criterion, in terms of one-parameter subgroups, for the Cohen-Macaulayness of R^G_χ (Theorem 1.2). This result allows us to prove a large part of Stanley's conjecture (see Theorem 1.3 below).

Let $\chi \in X(T)$, $\lambda \in X(T)^*$. Define

$$I_\lambda = \{i \in \{1, \dots, d\} \mid \langle\lambda, \alpha_i\rangle \geq 0\} \tag{2}$$

Then we say that χ is *good for* (T, W, λ) if χ is not of the form $\sum_{i=1}^d a_i\alpha_i$ where $(a_i)_{i=1,\dots,d}$ is a set of integers with the property that $a_i < 0$ if $i \in I_\lambda$ and $a_i \geq 0$ otherwise. We say that χ is *good for* (T, W) if χ is good for all (T, W, λ) with $\lambda \neq 0$.

Now let χ be an irreducible character of G, with corresponding highest weight χ_{hi}. Then we say that χ is *good for* (G, W) if for all $S \subset \Phi$, $\chi_{\text{hi}} - \sum_{\rho\in S}\rho$ is good for (T, W).

Denote $X = \operatorname{Spec} R$. Let us call a point on X *stable* if it has a closed orbit and finite stabilizer.

We prove the following result :

Theorem 1.2. *Assume that X has a G-stable point. Let χ be an irreducible character of G, good for (G, W). Then R^G_χ is Cohen-Macaulay.*

Note that the spirit of 1.1 and 1.2 is that for R^G_χ to be Cohen-Macaulay, χ should be "small" in comparison with W.

As a corollary to Theorem 1.2 we obtain the following result :

Theorem 1.3. *Assume that X has a G-stable point. Let χ be an irreducible character of G. Suppose that one of the following is true :*

(1) *χ is strongly critical for (G, W).*
(2) *χ is critical for (G, W) and every G-invariant subspace of codimension one in X has a T-stable point.*

Then R^G_χ is Cohen-Macaulay.

This theorem may be used in conjunction with the following observation :

Proposition 1.4. *Assume that G is semisimple and that X has a G-stable point. Let χ be an irreducible character of G such that $R^G_\chi \neq 0$. Then χ will be critical for (G, W) if and only if it is strongly critical for (G, W).*

The foregoing results will be proved in section 6.

Let $h = \dim R^G$ and let X^u be the null cone in X. Our approach, to study Cohen-Macaulayness of R^G_χ, will be based on the following lemma :

Lemma 1.5. [10] *R^G_χ is Cohen-Macaulay if and only if there is no representation with character χ in $H^i_{X^u}(X, \mathcal{O}_X)$ for $i = 0, \dots, h-1$.*

This lemma reduces the problem to the computation (or at least estimation) of $H^i_{X^u}(X, \mathcal{O}_X)$. To this end we develop a generalization of the well known stratification of X^u, due to Hesselink [2]. This, together with the observation that cohomology with support is a special case of relative algebraic De Rham homology, allows us to construct a spectral sequence which abuts to $H^i_{X^u}(X, \mathcal{O}_X)$. We then bound the terms in this spectral sequence.

This approach is inspired by the one used in [10]. However this paper is independent of [10]. On the other hand, essential use is made of the results in [9], which handled the torus case.

An interesting question remaining is whether our methods suffice to give a new proof of the Hochster-Roberts theorem. We may use the Luna-Richardson restriction theorem [4] to reduce to the case where X has a stable point. Furthermore we may assume that $\dim X^u > \dim G$ since otherwise it is well known that the problem is trivial. The question is now whether, under these hypothesis, the trivial representation is good for (G, W). I don't know this.

If G is a torus then the trivial representation is always (strongly) critical. This is not true in general however, as the following easy example shows.

Example 1.6. Let V be a 3-dimensional k-vectorspace. Define $G = \mathrm{Sl}(V)$ and $W = V^4$. Then X has a G-stable point and $\dim X^u > \dim G$. The trivial representation is not critical but it is easy to see that it is good. This shows that sometimes Theorem 1.3 is better than conj. 1.1.

This paper is organized as follows :
In section 2 we introduce some often used notations.
In section 3 we review algebraic De Rham homology.
In section 4 we develop a generalization of the classical stratification of X^u.
In section 5 we construct a spectral sequence, abutting to $H^i_{X^u}(X, \mathcal{O}_X)$.
In section 6 we apply the results of section 4 and 5 to give the proofs of Theorems 1.2 and 1.3.

2. Notations and conventions

In the sequel k will always be an algebraically closed field of characteristic zero.

If G is a linear algebraic group over k then $\mathcal{W}_G$ will be the Weylgroup of G. $Y(G)$ will be the pointed set of one-parameter subgroups of G.

If A is an abelian group then $A_{\mathbb{R}}$ will be defined as $\mathbb{R} \otimes_{\mathbb{Z}} A$.

If T is a torus over k then the character group of T will be denoted by $X(T)$ and the group law will be written additively. Since T is a torus, $Y(T)$ also carries an abelian group structure and there is a natural pairing $Y(T) \times X(T) \to X(G_m) \cong \mathbb{Z}$ given by composition. This pairing will be denoted by $\langle \ , \ \rangle$. We will extend this pairing to $Y(T)_{\mathbb{R}} \times X(T)_{\mathbb{R}}$. The pairing $\langle \ , \ \rangle$ allows one to identify $Y(T)$ with $X(T)^*$.

If $P \subset G$ is an algebraic subgroup of G and X is a scheme with a P-action then $G \times^P X = \overline{G \times X/P}$. There is a natural projection map $G \times^P X \to G/P$ given by $\overline{(g,x)} \mapsto \overline{g}$, with fibers isomorphic to X. Taking the fiber over $[P]$ in G/P induces an equivalence between the category of quasicoherent $\mathcal{O}_{G \times^P X}$-modules with a G-action and the category of quasicoherent $\mathcal{O}_X$-modules with a P-action. The inverse of this equivalence will be denoted by $\tilde{}$.

3. Algebraic De Rham homology

In this section we collect the results and definitions we need from [1]. The only thing new is the somewhat greater generality in which Tr_π and π_* are defined.

We will fix some base scheme S over k.

An embeddable S-scheme $\theta : T \to S$ will be an S-scheme such that the structure map θ factors as

$$\begin{array}{ccc} T & \hookrightarrow & X \\ {\scriptstyle\theta} & \searrow & \downarrow {\scriptstyle\theta} \\ & & S \end{array} \tag{3}$$

where the horizontal map is a closed immersion and the vertical map is smooth.

Let n be the relative dimension of X/S and let $\Omega^{\cdot}_{X/S}$ be the relative De Rham complex. Then the relative De Rham homology $H_i^{\mathrm{DR}}(T/S)$ is defined as $\mathbb{R}^{2n-i}\theta_{T_*}(\Omega^{\cdot}_{X/S})$. Here θ_{T_*} is the composition $\theta_* \underline{\underline{\Gamma}}_T$ (where $\underline{\underline{\Gamma}}_T$ denotes the *sheaf* of sections with support in T). It can be shown that this definition is independent of X.

We will use relative algebraic De Rham homology as a generalization of homology with support.

Lemma 3.1. *Assume that $\theta : T \to S$ is a closed immersion. Then*

$$H_{-i}^{\mathrm{DR}}(T/S) = H_T^i(S, \mathcal{O}_S)$$

Proof. Clear. □

In [1] the properties of relative algebraic De Rham homology are proved through the use of a canonical injective resolution $0 \to \Omega^{\cdot}_{X/S} \to E(\Omega^{\cdot}_{X/S})$ Then

$$H_i^{\mathrm{DR}}(T/S) = \mathcal{H}^{2n-i}(\theta_* \underline{\underline{\Gamma}}_T(E(\Omega^{\cdot}_{X/S})))$$

Relative algebraic De Rham homology is a covariant functor for proper maps. I.e. if $\pi : T' \to T$ is a proper map between embeddable S-schemes then there is a map $\pi_* : H_i^{\mathrm{DR}}(T'/S) \to H_i^{\mathrm{DR}}(T/S)$. The construction of π_* is somewhat involved and is explained below.

It is possible to construct a commutative diagram of S-schemes

$$\begin{array}{ccc} T' & \hookrightarrow & X' \\ \downarrow {\scriptstyle\pi} & & \downarrow {\scriptstyle\pi} \\ T & \hookrightarrow & X \end{array}$$

where the horizontal maps are closed immersions, X', X are smooth over S with structure maps θ', θ and $\pi : X' \to X$ is smooth. Let n',

n be the relative dimensions of X' and X over S. There is a certain canonically defined map, called the trace map

$$\mathrm{Tr}_\pi : \pi_* \underline{\underline{\Gamma}}_{T'}(E(\Omega^{\cdot}_{X'/S})[2n']) \to \underline{\underline{\Gamma}}_T(E(\Omega^{\cdot}_{X/S})[2n])$$

Applying the functor θ_* yields a map

$$\theta_*(\mathrm{Tr}_\pi) : \theta'_* \underline{\underline{\Gamma}}_{T'}(E(\Omega^{\cdot}_{X'/S})[2n']) \to \theta_* \underline{\underline{\Gamma}}_T(E(\Omega^{\cdot}_{X/S})[2n])$$

Then π_* is defined as the map induced on homology by $\theta_*(\mathrm{Tr}_\pi)$.

For a diagram

$$\begin{array}{ccc} T'' & \hookrightarrow & X'' \\ \downarrow & & \downarrow \pi' \\ T' & \hookrightarrow & X' \\ \downarrow & & \downarrow \pi \\ T & \hookrightarrow & X \end{array}$$

we have that

$$\mathrm{Tr}_{\pi \circ \pi'} = (\mathrm{Tr}_\pi) \circ (\pi_* \mathrm{Tr}_{\pi'}) \tag{4}$$

Taking homology yields that

$$(\pi \circ \pi')_* = \pi_* \circ \pi'_* \tag{5}$$

The following lemma will be used :

Lemma 3.2. *Assume that $\pi : T' \to T$ is a proper map between embeddable S-schemes, of finite type over the groundfield. Assume furthermore that π is settheoretically a bijection. Then π_* defines an isomorphism between $H_i^{\mathrm{DR}}(T'/S)$ and $H_i^{\mathrm{DR}}(T/S)$.*

Proof. We use induction on the dimension of T. The lemma is clearly correct if $\dim T = 0$. In general, since we are in characteristic zero, there will be a dense open $U \subset T$ such that π defines an isomorphism between $\pi^{-1}(U)$ and U. Therefore π_* defines isomorphisms between $H_i^{\mathrm{DR}}(\pi^{-1}(U)/S)$ and $H_i^{\mathrm{DR}}(U/S)$ and between $H_i^{\mathrm{DR}}(T' - \pi^{-1}(U)/S)$ and $H_i^{\mathrm{DR}}(T - U/S)$ (induction). The lemma now follows from the five lemma and the relative version of [1, Th 3.3]. $\square$

We will have to define Tr_π and π_* in a slightly more general situation : Assume that there is a commutative diagram

$$\begin{array}{ccccc} Y' & \hookrightarrow & T' & \hookrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \pi \\ Y & \hookrightarrow & T & \hookrightarrow & X \end{array}$$

where the horizontal maps are closed immersions and π is a smooth map between smooth S-schemes X', X with relative dimension n', n over S.

Let $U' = T' - Y'$, $U = T - Y$ and denote the injections $X' - Y' \hookrightarrow X'$, $X - Y \hookrightarrow X$ by i' and i. Then we may construct a commutative diagram (6)

$$\begin{array}{ccccccccc}
0 & \to & \pi_* \underline{\underline{\Gamma}}_{Y'}(E(\Omega^\cdot_{X'/S})[2n']) & \to & \pi_* \underline{\underline{\Gamma}}_{T'}(E(\Omega^\cdot_{X'/S})[2n']) & \to & \pi_* i'_* \underline{\underline{\Gamma}}_{U'}(E(\Omega^\cdot_{X'-Y'/S})[2n']) & \to & 0 \\
 & & \downarrow \mathrm{Tr}_\pi & & \downarrow \mathrm{Tr}_\pi & & \downarrow \mathrm{F} & & \\
0 & \to & \underline{\underline{\Gamma}}_{Y}(E(\Omega^\cdot_{X/S})[2n]) & \to & \underline{\underline{\Gamma}}_{T}(E(\Omega^\cdot_{X/S})[2n]) & \to & i_* \underline{\underline{\Gamma}}_{U'}(E(\Omega^\cdot_{X-Y/S})[2n]) & \to & 0
\end{array}$$

Assume now that we have another diagram

$$\begin{array}{ccccc}
Y'_1 & \hookrightarrow & T'_1 & \hookrightarrow & X' \\
\downarrow & & \downarrow & & \downarrow \pi \\
Y_1 & \hookrightarrow & T_1 & \hookrightarrow & X
\end{array}$$

where $Y'_1 \subset Y'$, $T'_1 \subset T'$, $Y_1 \subset Y$, $T_1 \subset T$ and $T'_1 - Y'_1 = U'$, $T_1 - Y_1 = U$. The injections $X' - Y'_1 \hookrightarrow X'$, $X - Y_1 \hookrightarrow X$ will now be denoted by i'_1 and i_1. Using a similar diagram as (6) we may construct a map

$$\mathrm{F}_1 : \pi_* i'_{1*} \underline{\underline{\Gamma}}_{U'}(E(\Omega^\cdot_{X-Y'_1/S})[2n']) \to \pi_* i_{1*} \underline{\underline{\Gamma}}_{U}(E(\Omega^\cdot_{X-Y_1/S})[2n])$$

Combining the diagrams for F and F_1 yields a commutative square

$$\begin{array}{ccc}
\pi_* i'_{1*} \underline{\underline{\Gamma}}_{U'}(E(\Omega^\cdot_{X-Y'_1/S})[2n']) & \to & \pi_* i'_* \underline{\underline{\Gamma}}_{U'}(E(\Omega^\cdot_{X-Y'/S})[2n']) \\
\downarrow \mathrm{F}_1 & & \downarrow \mathrm{F} \\
\pi_* i_{1*} \underline{\underline{\Gamma}}_{U}(E(\Omega^\cdot_{X-Y_1/S})[2n]) & \to & \pi_* i_* \underline{\underline{\Gamma}}_{U}(E(\Omega^\cdot_{X-Y/S})[2n])
\end{array}$$

where the horizontal maps are obtained from the injections $X - Y' \hookrightarrow X - Y'_1$, $X - Y \hookrightarrow X - Y_1$. Hence these horizontal maps are isomorphisms. This means that, in an appropriate sense, F depends only on the data (U, U', X, X', π) and not on the particular choice of (T, T'). Therefore F will be denoted by Tr_π in this more general setting, even if π does not restrict to a map $U' \to U$.

Tr_π still satisfies the compatibility condition for compositions of maps (4).

Let θ be the structure map of X. Applying the functor θ_* to F and taking homology, yields a map

$$H_i^{\mathrm{DR}}(U'/S) \to H_i^{\mathrm{DR}}(U/S)$$

which we will also denote by π_*. It is easy to see that (5) still holds.

4. The stratifications

Let G be a connected reductive group over k and let T, B be resp. a maximal torus in G and a Borel subgroup containing T. Denote by Φ the set of roots of G.

We will choose a positive definite, $\mathcal{W}_G$ invariant, quadratic form $(\ ,\ )$ on $Y(T)_{\mathbb{R}}$. The corresponding norm will be denoted by $\|\ \|$. $Y(T)_{\mathbb{R}}$ will be partially ordered by putting $\lambda < \lambda'$ if $\|\lambda\| < \|\lambda'\|$.

W will be a finite dimensional G-representation. Let $w_1, \ldots, w_d$ be a basis of W for which the action of T is diagonal with corresponding weights $\alpha_1, \ldots, \alpha_d \in X(T)$.

Let $R = SW$ and $X = \operatorname{Spec} R$. The closed points of X correspond to the elements of W^* and hence X is a linear space, spanned by the dual basis $w_1^*, \ldots, w_d^*$, on which T acts with weights $-\alpha_1, \ldots, -\alpha_d$.

For $\lambda \in X(T)$, define

$$X_\lambda = \{x \in X \mid \lim_{t \to 0} \lambda(t)x = 0\}$$

$$P_\lambda = \{g \in G \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}$$

Clearly $P_\lambda X_\lambda = X_\lambda$. Furthermore, it follows from [5, Prop. 2.6] that P_λ is a parabolic subgroup of G.

It is easy to see that X_λ is a linear subspace spanned by those w_i^* such that $\langle \lambda, \alpha_i \rangle < 0$. P_λ is the subgroup of G containing T and having roots $\rho \in \Phi$ such that $\langle \lambda, \rho \rangle \geq 0$. These descriptions still make sense for $\lambda \in Y(T)_{\mathbb{R}}$. Hence the notations X_λ and P_λ will also be used in this more general setting. It is still true that P_λ is parabolic and $P_\lambda X_\lambda = X_\lambda$.

If $\lambda \in Y(T)_{\mathbb{R}}$ then we define Y_λ to be the linear subspace of X spanned by those w_i^* such that $\langle \lambda, \alpha_i \rangle \leq -1$. By going to the Lie algebra we see that $P_\lambda Y_\lambda = Y_\lambda$. Also $X_\lambda = Y_{n\lambda}$ for $n \gg 0$.

If $U \subset Y(T)_{\mathbb{R}}$ then we define $X_U = \bigcup_{\lambda \in U} X_\lambda$. If P is a parabolic subgroup of G, containing T then

$$A_P = \{\lambda \in Y(T)_{\mathbb{R}} \mid P_\lambda \supset P\}$$

I.e.

$$A_P = \{\lambda \in Y(T)_{\mathbb{R}} \mid \langle \lambda, \rho \rangle \geq 0 \text{ for all roots } \rho \text{ of } P\}$$

Note that any element of $Y(T)_{\mathbb{R}}$ is conjugate to a unique element of A_B.

X_P will be defined as X_{A_P}. Using this notation, the Hilbert Mumford criterion may be written as

$$X^u = G X_B \tag{7}$$

In this section we seek to stratify PX_B for P a parabolic subgroup G containing B. This stratification will be the basic technical tool in the next section.

The stratification of $X^u = GX_B$ corresponds to that of [2] and [3]. See however remark 4.5 below.

T defines an apartment in the Tits building defined by G. This is a Coxeter complex in the sense of [8]. We will use the geometrical realization from [6] of this complex.

The faces are the sets $A_P \subset Y(T)_{\mathbb{R}}$ defined above. The faces of maximal dimension (the "chambers") are those A_P where P is a Borel subgroup of G.

Cham St P is defined to be the union of those chambers containing A_P. This is a convex set according to [8, prop. 2.7] and [6]. If $P \supset B$ then it follows from [8, p. 17] that

$$\text{Cham St}\, P = \bigcup_{w \in \mathcal{W}_P} A_B$$

If $\lambda \in Y(T)_{\mathbb{R}}$ then Cham St P_λ is the union of the chambers containing λ. Hence λ is an interior point of Cham St P_λ.

If P, $P' \supset B$ are parabolic subgroups of G then it is easy to see that the sets Cham St $P \cap$ Cham St P' and Cham St $P \cap P'$ contain the same chambers. Since Cham St $P \cap$ Cham St P' is convex complex, containing the chamber A_B, it must be a union of chambers [8, cor. 2.21].

Therefore we deduce

$$\text{Cham St}\, P \cap \text{Cham St}\, P' = \text{Cham St}\, P \cap P'$$

Now we start with some preparatory work

Lemma 4.1. *Let λ, λ', $\lambda'' \in Y(T)_{\mathbb{R}}$ such that $\lambda'' \in [\lambda, \lambda']$. Then*

$$Y_\lambda \cap Y_{\lambda'} \subset Y_{\lambda''}$$

Proof. $Y_\lambda \cap Y_{\lambda'}$ is spanned by those w_i^* such that $\langle \lambda, \alpha_i \rangle \leq -1$, $\langle \lambda', \alpha_i \rangle \leq -1$. This implies $\langle \lambda'', \alpha_i \rangle \leq -1$. $\square$

This lemma will be used in the following setting

Lemma 4.2. *Assume that $D \subset D'$ are convex subsets of $Y(T)_{\mathbb{R}}$. Let $\lambda \in D$, $\lambda' \in D'$, $\|\lambda'\| \leq \|\lambda\|$, $\lambda \neq \lambda'$ and assume that λ is in the relative interior of D in D'. Then there exists a $\lambda'' \in D$ such that $\lambda'' < \lambda$ and $Y_\lambda \cap Y_{\lambda'} \subset Y_{\lambda''}$.*

Proof. Our hypothesis imply that there exists a $\lambda'' \in]\lambda, \lambda'] \cap D$. Using lemma 4.1, it is easy to see that λ'' has the required properties. $\square$

Lemma 4.3. *Let $E \subset X$. Then the set*

$$\{\lambda \in A_B \mid E \subset Y_\lambda\} \tag{8}$$

is closed convex and hence, if it is non empty, it contains a unique minimal element.

Proof. Standard, using the definition of Y_λ, and lemma 4.1 $\square$

Let $\mathcal{B}$ be the set of elements of A_B that occur as minimal elements of sets of the form (8).

Lemma 4.4. *$\mathcal{B}$ is a finite set.*

Proof. Define

$$S_E = \{i \in \{1, \dots, d\} \mid \exists x \in E, x_i \neq 0\}$$

Then the set (8) is equal to the set

$$\{\lambda \in A_B \mid \langle \lambda, \alpha_i \rangle \leq -1, \forall i \in S_E\}$$

Since there are only a finite number of possibilities for S_E, $\mathcal{B}$ must be a finite set. $\square$

Remark 4.5. $\mathcal{B}$ will be the indexing set for our stratifications of the sets PX_B. This indexing set does not correspond completely to the one used in [2] and [3]. In particular a stratum that will be indexed by λ in our sense, will be indexed by $\lambda/\|\lambda\|^2$ in [2] and [3].

Furthermore there are elements in $\mathcal{B}$ that do not index strata in $X^u = GX_B$ in the sense of [2] and [3]. It turns out that our strata are empty in this case. However this is not necessarily true for the corresponding strata in PX_B, $P \neq G$.

Lemma 4.6. *Let $\lambda \in A_B$. Then there exists a $\lambda' \in \mathcal{B}$, $\|\lambda'\| \leq \|\lambda\|$ such that $Y_\lambda \subset Y_{\lambda'}$*

Proof. Clear from the definition of $\mathcal{B}$. $\square$

Lemma 4.7. *Let $\lambda \in A_B$ and let P be a parabolic subgroup of G, containing B. Then PY_λ, PX_λ are closed in X.*

Proof. This follows, in a standard way, from the fact that there is a Borel subgroup B stabilizing Y_λ and X_λ. $\square$

If P is a parabolic subgroup of G, containing B and $\lambda \in \mathcal{B}$ then we define

$$S_{P,\lambda} = PY_\lambda - \bigcup_{\substack{\lambda' < \lambda \\ \lambda' \in \mathcal{B}}} PY_{\lambda'}$$

Lemma 4.8. *Assume that P, P' are parabolic subgroups of G such that P, $P' \subset P_\lambda$ for some $\lambda \in Y(T)_{\mathbb{R}}$. Then $S_{P,\lambda} \subset X_{P'}$*

Proof. Since $X_{P'} \supset X_{P_\lambda}$ it is sufficient to show that $S_{P,\lambda} \subset X_{P_\lambda}$. Now, since X_{P_λ} is P-invariant and $Y_\lambda \subset X_\lambda$, this follows from $X_\lambda \subset X_{P_\lambda}$ which is implied by the definition of X_{P_λ}. □

Proposition 4.9. *Let $\mathcal{C} \subset \mathcal{B}$ be a set with the property that $\lambda \in \mathcal{C}$, $\lambda' \in \mathcal{B}$, $\lambda' < \lambda$ implies $\lambda' \in \mathcal{C}$. Then*

$$\bigcup_{\lambda \in \mathcal{C}} S_{P,\lambda} = \bigcup_{\lambda \in \mathcal{C}} PY_\lambda$$

Proof. Choose $y \in PY_\lambda$, $\lambda \in \mathcal{C}$ and let λ' be a minimal element of $\mathcal{B}$ such that $\lambda' \leq \lambda$ and $y \in PY_{\lambda'}$. Clearly $\lambda' \in \mathcal{C}$ and $y \in S_{P,\lambda'}$. □

Proposition 4.10.

(1) $\bigcup_{\lambda \in \mathcal{B}} S_{P,\lambda} = PX_B$
(2) *If λ, $\lambda' \in \mathcal{B}$, $\lambda \neq \lambda'$ then $S_{P,\lambda} \cap S_{P,\lambda'} = \emptyset$*

Proof. (1) Using lemma 4.6 and prop. 4.9 we compute :

$$\bigcup_{\lambda \in \mathcal{B}} S_{P,\lambda} = \bigcup_{\lambda \in \mathcal{B}} PY_\lambda = \bigcup_{\lambda \in A_B} PY_\lambda$$

$$= \bigcup_{\lambda \in A_B} PX_\lambda = PX_B$$

(2) By symmetry we may assume that $\|\lambda'\| \leq \|\lambda\|$. It is sufficient to prove that

$$PY_\lambda \cap PY_{\lambda'} \subset \bigcup_{\lambda'' < \lambda} PY_{\lambda''}$$

Using Bruhat's lemma for P, this follows from

$$\forall w' \in \mathcal{W}_P, \exists w'' \in \mathcal{W}_P, \exists \lambda'' \in A_B, \lambda'' < \lambda : Y_\lambda \cap w'Y_{\lambda'} \subset w''Y_{\lambda''} \tag{9}$$

Let $\lambda_1' = w'\lambda'$. Then $\lambda_1' \in \text{Cham St}\, P$ and $\lambda_1' \neq \lambda$.

To prove (9) it is sufficient to show that there exists a $\lambda_1'' \in \text{Cham St}\, P$, $\lambda_1'' < \lambda$ such that

$$Y_\lambda \cap Y_{\lambda_1'} \subset Y_{\lambda_1''}$$

This follows from lemma 4.2, with D and D' both equal to $\text{Cham St}\, P$.

□

Proposition 4.11. $\overline{S}_{P,\lambda} \subset \bigcup_{\|\lambda'\| \leq \|\lambda\|} S_{P,\lambda}$

Proof. Clearly $\overline{S}_{P,\lambda} \subset PY_\lambda$. But by prop. 4.9

$$\bigcup_{\|\lambda'\| \leq \|\lambda\|} S_{P,\lambda} = \bigcup_{\|\lambda'\| \leq \|\lambda\|} PY_{\lambda'} \supset PY_\lambda$$

□

Proposition 4.12. *Let $\lambda \in A_B$, and assume that P is a parabolic subgroup of G containing B. Then*

(1) $PS_{P_\lambda \cap P, \lambda} = S_{P,\lambda}$

(2) *The natural map*

$$\pi : P \times^{P_\lambda \cap P} S_{P_\lambda \cap P, \lambda} \to S_{P,\lambda}$$

is settheoretically a bijection.

Proof. (1) We have to show that

$$P(Y_\lambda - \bigcup_{\substack{\lambda'' < \lambda \\ \lambda'' \in \mathcal{B}}} (P_\lambda \cap P) Y_{\lambda''}) = P(Y_\lambda - \bigcup_{\substack{\lambda' < \lambda \\ \lambda' \in \mathcal{B}}} PY_{\lambda'})$$

which would follow from

$$Y_\lambda - \bigcup_{\substack{\lambda'' < \lambda \\ \lambda'' \in A_B}} (P_\lambda \cap P) Y_{\lambda''} = Y_\lambda - \bigcup_{\substack{\lambda' < \lambda \\ \lambda' \in A_B}} PY_{\lambda'} \tag{10}$$

For this it is sufficient that for any $\lambda' \in A_B$, $\lambda' < \lambda$

$$Y_\lambda \cap PY_{\lambda'} \subset \bigcup_{\substack{\lambda'' < \lambda \\ \lambda'' \in A_B}} (P_\lambda \cap P) Y_{\lambda''}$$

Fix λ'. Using Bruhat's lemma for P and $P_\lambda \cap P$, it is sufficient that for any $w' \in \mathcal{W}_P$ there are $w'' \in \mathcal{W}_{P_\lambda \cap P}$, $\lambda'' \in A_B$, $\lambda'' < \lambda$ such that

$$Y_\lambda \cap w' Y_{\lambda'} \subset w'' Y_{\lambda''}$$

Put $\lambda'_1 = w'\lambda'$. Then $\lambda'_1 \in \text{Cham St}\, P$. We have to find a $\lambda''_1 \in \text{Cham St}\, P \cap P_\lambda$, $\lambda''_1 < \lambda$ such that

$$Y_\lambda \cap Y_{\lambda'_1} \subset Y_{\lambda''_1}$$

This follows from lemma 4.2 provided λ lies in the relative interior of Cham St $P \cap P_\lambda$ in Cham St P. This follows from the fact that $\lambda \in (\text{Cham St}\, P_\lambda)^\circ$ and $\text{Cham St}\, P \cap P_\lambda = \text{Cham St}\, P \cap \text{Cham St}\, P_\lambda$.

(2) Every element in $S_{P,\lambda} = PY_\lambda - \bigcup_{\substack{\lambda' < \lambda \\ \lambda' \in A_B}} PY_{\lambda'}$ lies in the P-orbit of some element in $Y_\lambda - \bigcup_{\substack{\lambda' < \lambda \\ \lambda' \in A_B}} PY_{\lambda'}$. Hence let $y \in$

$Y_\lambda - \bigcup_{\substack{\lambda'<\lambda\\ \lambda'\in A_B}} PY_{\lambda'}$. Then one computes ($\cong$ means "there is a bijection").

$$\begin{aligned}
\pi^{-1}(y) &= \Big\{(p,y')\in P\times (Y_\lambda - \bigcup_{\substack{\lambda''<\lambda\\ \lambda''\in A_B}} (P_\lambda\cap P)Y_{\lambda''}) \mid py'=y\Big\}/P_\lambda\cap P\\
&\cong \Big\{p\in P\mid p^{-1}y\in Y_\lambda - \bigcup_{\substack{\lambda''<\lambda\\ \lambda''\in A_B}} (P_\lambda\cap P)Y_{\lambda''}\Big\}/P_\lambda\cap P\\
&\cong \Big\{p\in P\mid p^{-1}y\in Y_\lambda - \bigcup_{\substack{\lambda'<\lambda\\ \lambda'\in A_B}} PY_{\lambda'}\Big\}/P_\lambda\cap P \quad \text{(using (10))}\\
&\cong \Big\{p\in P\mid y\in pY_\lambda - \bigcup_{\substack{\lambda'<\lambda\\ \lambda'\in A_B}} PY_{\lambda'}\Big\}/P_\lambda\cap P\\
&\cong \Big\{p\in P\mid y\in pY_\lambda\Big\}/P_\lambda\cap P \quad \text{(using the definition of } y) \qquad (11)
\end{aligned}$$

Now (11) will be a singleton provided that

$$Y_\lambda\cap (P-P\cap P_\lambda)Y_\lambda\subset \bigcup_{\substack{\lambda'<\lambda\\ \lambda'\in\mathcal{B}}} PY_{\lambda'}$$

Again, using Bruhat's lemma, it is sufficient to show that for any $w\in\mathcal{W}_P-\mathcal{W}_{P\cap P_\lambda}$ there exist $\lambda''\in A_B$, $\lambda''<\lambda$ and $w''\in\mathcal{W}_P$ such that

$$Y_\lambda\cap wY_\lambda\subset w''Y_{\lambda''}$$

Put $\lambda'=w\lambda$. Then $\|\lambda'\|=\|\lambda\|$ but $\lambda\neq\lambda'$. Otherwise $w\in\mathcal{W}_{P_\lambda}$ and since $w\in\mathcal{W}_P$ this implies $w\in\mathcal{W}_{P\cap P_\lambda}$, which was excluded.

We have to find $\lambda''_1\in \operatorname{Cham}\operatorname{St}P$, $\lambda''_1<\lambda$ such that

$$Y_\lambda\cap Y_{\lambda'}\subset Y_{\lambda''_1}$$

This follows from lemma 4.2, if we put $D=D'=\operatorname{Cham}\operatorname{St}P$. $\square$

Remark 4.13. It seems quite likely that π is actually an isomorphism (if $G=P$ then this follows from the results in [2][3]). We have not proved this since it will not be needed in the sequel.

5. The main step

5.1. **Some definitions.** In this section, G will be a connected reductive algebraic group with a Borel subgroup denoted by B.

If $P \supset Q$ are parabolic subgroups in G containing B and if there is a maximal chain

$$Q = P_0 \subset \cdots \subset P_u = P$$

then u will be denoted by $l(P/Q)$. $l(G/B)$ will be denoted by r. Define

$$\mathcal{Q} = \{\text{parabolic subgroups of } G, \text{ containing } B\}$$

If Q, $Q' \in \mathcal{Q}$ then we say that Q is a face of Q', if $Q \subset Q'$.

It is easy to see that this definition makes $\mathcal{Q}$ into an abstract complex in the sense of [8]. The faces of dimension n in $\mathcal{Q}$ are given by

$$\mathcal{Q}_n = \{Q \in \mathcal{Q} \mid l(Q/B) = n+1\}$$

The boundary maps $\partial : \mathbb{Z}\mathcal{Q}_n \to \mathbb{Z}\mathcal{Q}_{n-1}$ define a set of numbers $\alpha_{P,Q} \in \{\pm 1, 0\}$

$$\partial(Q) = \sum \alpha_{Q,Q'} Q'$$

Fix $R_1 \subset R_2$, parabolic subgroups of G, containing B. Put $l(R_1/B) = u$, $l(R_2/B) = r'$ and define

$$\mathcal{Q}' = \{Q \in \mathcal{Q} \mid R_1 \subset Q \subset R_2\}$$

$$\mathcal{Q}'_n = \mathcal{Q}' \cap \mathcal{Q}_{n+u}$$

Then $\mathcal{Q}'$ is again an abstract complex. The corresponding boundary maps $\partial' : \mathbb{Z}\mathcal{Q}'_n \to \mathbb{Z}\mathcal{Q}'_{n-1}$ are now given by

$$\partial'(Q) = \sum_{Q' \in \mathcal{Q}'} \alpha_{Q,Q'} Q'$$

If $r' > 0$ then $\mathcal{Q}'$ is a combinatorial simplex. Hence the reduced chain complex

$$0 \to \mathbb{Z}\mathcal{Q}'_{r'-1} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \mathbb{Z}\mathcal{Q}'_{-1} \to 0 \tag{12}$$

will be acyclic. Note however that if $r' = 0$ then (12) is *not* acyclic.

We will need one more abstract complex. Define

$$\mathcal{R} = \{(P,Q) \in \mathcal{Q} \times \mathcal{Q} \mid Q \subset P\}$$

$$\mathcal{R}_n = \{(P,Q) \in \mathcal{R} \mid l(P/Q) + n = r - 1\}$$

We let (P,Q) be a face of (P',Q') if $P \supset P'$, $Q \subset Q'$. This makes $\mathcal{R}$ into an abstract complex whose corresponding topological space is a $r-1$-dimensional sphere.

We now define numbers $\alpha_{(P,Q),(P',Q')}$, β_Q as follows :

$$\alpha_{(P,Q),(P',Q')} = \begin{cases} \alpha_{P',P}(-)^{l(Q/B)} & \text{if } l(P'/P)=1,\ Q=Q' \\ \alpha_{Q,Q'} & \text{if } l(Q/Q')=1,\ P=P' \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

$$\beta_Q = (-)^{\lceil \frac{l(Q/B)}{2} \rceil} \tag{14}$$

Lemma 5.1.1.

(1) *Let* $(P,Q) \in \mathcal{R}_n$, $(P'',Q'') \in \mathcal{R}_{n-2}$ *such that* (P'',Q'') *is a face of* (P,Q). *Assume that* (P_1',Q_1'), $(P_2',Q_2') \in \mathcal{R}_{n-1}$ *are the two faces of* (P,Q) *having* (P'',Q'') *as a face. Then*

$$\alpha_{(P,Q),(P_1',Q_1')}\alpha_{(P_1',Q_1'),(P'',Q'')} + \alpha_{(P,Q),(P_2',Q_2')}\alpha_{(P_2',Q_2'),(P'',Q'')} = 0$$

(2) *Assume that* $(P,Q) \in \mathcal{R}_{r-2}$. *Then*

$$\beta_Q \alpha_{(Q,Q),(P,Q)} + \beta_P \alpha_{(P,P),(P,Q)} = 0$$

Proof. This follows by inspection. □

5.2. **A spectral sequence.** In this section G, B, T, X will have the same meaning as in section 4. $l(G/B)$ will still be denoted by r.

We will construct a spectral sequence that will allow us to bound $H^i_{X^u}(X,\mathcal{O}_X)$. To this end we use algebraic De Rham homology with base scheme X. This idea is inspired by lemma 3.1.

If $P_2 \supset P_1$ are parabolic subgroups of G then $\pi^{P_1}_{P_2}$ will be used for the canonical maps $G/P_1 \to G/P_2$, $G \times^{P_1} X \to G \times^{P_2} X$. If $T_1 \subset G \times^{P_1} X$, $T_2 \subset G \times^{P_2} X$ are subschemes such that $\pi^{P_1}_{P_2}(T_1) \subset T_2$ then the map $T_1 \to T_2$, which is the restriction of $\pi^{P_1}_{P_2}$ will still be denoted by $\pi^{P_1}_{P_2}$.

If we are in the more general situation of (6)

$$\begin{array}{ccccc} Y_1 & \hookrightarrow & T_1 & \hookrightarrow & G \times^{P_1} X \\ \downarrow & & \downarrow & & \downarrow \pi^{P_1}_{P_2} \\ Y_2 & \hookrightarrow & T_2 & \hookrightarrow & G \times^{P_2} X \end{array} \tag{15}$$

where $U_2 = T_2 - Y_2$, $U_1 = T_1 - Y_1$. Then

$$\pi^{P_1}_{P_2 *} : H^{\mathrm{DR}}_i(U_1/X) \to H^{\mathrm{DR}}_i(U_2/X)$$

and the corresponding trace maps are defined as in section 4.

Theorem 5.2.1. *There is a second quadrant spectral sequence*

$$E^1_{-p,q} : \oplus_{(P,Q)\in\mathcal{R}_{r-1-p}} H^{\mathrm{DR}}_{-q}(G \times^Q X_P/X) \Rightarrow H^{-p+q}_{X^u}(X,\mathcal{O}_X) \tag{16}$$

The differentials $d : E^1_{-p,q} \to E^1_{-p+1,q}$ *are given by*

$$\oplus \alpha_{(P',Q'),(P,Q)} \pi^Q_{Q'*} \tag{17}$$

where the sum runs over all pairs

$$((P,Q),(P',Q')) \in \mathcal{R}_{r-1-p} \times \mathcal{R}_{r-p}$$

The proof of this theorem will be based on lemma 5.2.2 below.

Let $\mathcal{C}$ be a subset of $\mathcal{B}$ as in prop. 4.9 and define

$$T = \bigcup_{\lambda \in \mathcal{C}} S_{B,\lambda} = \bigcup_{\lambda \in \mathcal{C}} Y_\lambda$$

For P, $Q \in \mathcal{Q}$ put,

$$T_{P,Q} = QT \cap QX_P$$

$$E^{\cdot}_{P,Q} = \underline{\Gamma}_{G\times^Q T_{P,Q}}(E(\Omega^{\cdot}_{G\times^Q X/X})[2\dim G/Q])$$

$T_{P,Q}$ is a closed subset of X (lemma 4.7) and $\pi^Q_{G*}E^{\cdot}_{P,Q}$ is a complex of flasque sheaves on X whose homology in degree q is

$$H^{\mathrm{DR}}_{-q}(G \times^Q T_{P,Q}/X) \tag{18}$$

Now we construct maps of complexes

$$\oplus_{(P,Q)\in\mathcal{R}_{r-1-p}}\pi^Q_{G*}E^{\cdot}_{P,Q} \xrightarrow{d} \oplus_{(P,Q)\in\mathcal{R}_{r-p}}\pi^Q_{G*}E^{\cdot}_{P,Q}$$

as follows : If $(P,Q) \in \mathcal{R}_{r-1-p}$, $(P',Q') \in \mathcal{R}_{r-p}$, $Q \subset Q'$, $P \supset P'$ then the canonical map $\pi^Q_{Q'} : G \times^Q X \to G \times^{Q'} X$ restricts to a map $G \times^Q T_{P,Q} \to G \times^{Q'} T_{P',Q'}$ and hence there is a trace map

$$\mathrm{Tr}_{\pi^Q_{Q'}} : \pi^Q_{Q'*}E^{\cdot}_{P,Q} \to E^{\cdot}_{P',Q'}$$

Applying the functor $\pi^{Q'}_{G*}$ yields a map

$$\pi^{Q'}_{G*}(\mathrm{Tr}_{\pi^Q_{Q'}}) : \pi^Q_{G*}E^{\cdot}_{P,Q} \to \pi^{Q'}_{G*}E^{\cdot}_{P',Q'}$$

Then d is given by

$$\oplus\alpha_{(P',Q'),(P,Q)}\pi^{Q'}_{G*}(\mathrm{Tr}_{\pi^Q_{Q'}}) \tag{19}$$

where the sum runs over all pairs

$$((P,Q),(P',Q')) \in \mathcal{R}_{r-1-p} \times \mathcal{R}_{r-p}$$

We also construct a map

$$\oplus_{(Q,Q)\in\mathcal{R}_{r-1}}\pi^Q_{G*}E^{\cdot}_{Q,Q} \xrightarrow{\epsilon} E^{\cdot}_{B,G}$$

as

$$\oplus_{(Q,Q)\in\mathcal{R}_{r-1}}\beta_Q \mathrm{Tr}_{\pi^Q_G} \tag{20}$$

Lemma 5.2.2. *With notations as above, there is a double complex*

(21)

$$\cdots \xrightarrow{d} \oplus_{(P,Q)\in\mathcal{R}_{r-1-p}} \pi^Q_{G*} E^\cdot_{P,Q} \xrightarrow{d} \oplus_{(P,Q)\in\mathcal{R}_{r-p}} \pi^Q_{G*} E^\cdot_{P,Q} \xrightarrow{d} \cdots \xrightarrow{\epsilon} E^\cdot_{B,G} \to 0$$

whose associated complex is exact.

Proof. That (21) is a double complex follows from (4) and lemma 5.1.1.

The rest of the proof will be by induction $|\mathcal{C}|$, the case $\mathcal{C} = \emptyset$ being trivial.

Assume that λ is a maximal element of $\mathcal{C}$ and define $T' = T - S_{B,\lambda}$, $T'_{P,Q} = QT' \cap QX_P$. It is clear from prop. 4.9 that $T_{P,Q}$ is the disjoint union of $T'_{P,Q}$ and $S_{Q,\lambda} \cap QX_P$. Let $i : G \times^Q (X - T'_{P,Q}) \hookrightarrow G \times^Q X$ be the open immersion and define

$$E^\cdot_{P,Q,\lambda} = i_* \underline{\Gamma}_{G\times^Q(S_{Q,\lambda}\cap QX_P)}(E(\Omega^\cdot_{G\times^Q(X-T'_{P,Q})/X})[2\dim G/Q])$$

Furthermore let $E'^\cdot_{P,Q}$ be defined as $E^\cdot_{P,Q}$, but using $T'_{P,Q}$ instead of $T_{P,Q}$.

Then, by induction, the lemma is true for $E'^\cdot_{P,Q}$. Furthermore there are exact sequences of complexes

$$0 \to E'^\cdot_{P,Q} \to E^\cdot_{P,Q} \to E^\cdot_{P,Q,\lambda} \to 0$$

Hence it is sufficient to show that the complex, obtained from the double complex

(22)

$$\cdots \xrightarrow{d} \oplus_{(P,Q)\in\mathcal{R}_{r-1-p}} \pi^Q_{G*} E^\cdot_{P,Q,\lambda} \xrightarrow{d} \oplus_{(P,Q)\in\mathcal{R}_{r-p}} \pi^Q_{G*} E^\cdot_{P,Q,\lambda} \xrightarrow{d} \cdots \xrightarrow{\epsilon} E^\cdot_{B,G,\lambda} \to 0$$

is exact.

Here d and ϵ are defined by (19) and (20). However we have to use the more general definition of the trace map, as in (15).

To show that (22) gives rise to an exact sequence, it is sufficient to show that it induces exact sequences on homology. I.e. we have to show that the complexes

$$\cdots \xrightarrow{d} \oplus_{(P,Q)\in\mathcal{R}_{r-1-p}} H^{\mathrm{DR}}_{-q}(G \times^Q (S_{Q,\lambda} \cap X_P)/X) \xrightarrow{d}$$

$$\oplus_{(P,Q)\in\mathcal{R}_{r-p}} H^{\mathrm{DR}}_{-q}(G \times^Q (S_{Q,\lambda} \cap X_P)/X) \xrightarrow{d}$$

$$\cdots \xrightarrow{\epsilon} H^{\mathrm{DR}}_{-q}(S_{G,\lambda}/X) \to 0 \tag{23}$$

are exact. (We have used that $QX_P = X_P$ if $Q \subset P$.)

d is still defined as in (17) and ϵ is given by

$$\oplus_{(P,Q)\in\mathcal{R}_{r-1}} \beta_Q \pi^Q_{G*}$$

$\pi_Q^{P_\lambda \cap Q}$ defines settheoretically a bijection (prop. 4.12)

$$G \times^{P_\lambda \cap Q} S_{P_\lambda \cap Q,\lambda} \to G \times^Q S_{Q,\lambda}$$

and by restricting $\pi_Q^{P_\lambda \cap Q}$ to the inverse image of $G \times^Q X_P$ we obtain a bijection

$$G \times^{P_\lambda \cap Q} (S_{P_\lambda \cap Q,\lambda} \cap X_P) \to G \times^Q (S_{Q,\lambda} \cap X_P)$$

Hence $\pi_{Q*}^{P_\lambda \cap Q}$ induces an isomorphism (lemma 3.2)

$$H^{\mathrm{DR}}_{-q}(G \times^{P_\lambda \cap Q} (S_{P_\lambda \cap Q,\lambda} \cap X_P)/X) \to H^{\mathrm{DR}}_{-q}(G \times^Q (S_{Q,\lambda} \cap X_P)/X)$$

We may therefore replace (23) by a complex

$$\cdots \xrightarrow{d} \oplus_{(P,Q)\in\mathcal{R}_{r-1-p}} H^{\mathrm{DR}}_{-q}(G \times^{P_\lambda \cap Q} (S_{P_\lambda \cap Q,\lambda} \cap X_P)/X) \xrightarrow{d}$$

$$\oplus_{(P,Q)\in\mathcal{R}_{r-p}} H^{\mathrm{DR}}_{-q}(G \times^{P_\lambda \cap Q} (S_{P_\lambda \cap Q,\lambda} \cap X_P)/X) \xrightarrow{d}$$

$$\cdots \xrightarrow{\epsilon} H^{\mathrm{DR}}_{-q}(G \times^{P_\lambda} S_{P_\lambda,\lambda}/X) \tag{24}$$

d is now given by a sum

$$\oplus \alpha_{(P',Q'),(P,Q)} \pi^{P_\lambda \cap Q}_{P_\lambda \cap Q' *}$$

and ϵ is given by

$$\oplus \beta_Q \pi^{P_\lambda \cap Q}_{P_\lambda *}$$

(24) has a subcomplex given by

(25)

$$\cdots \xrightarrow{d} \oplus_{\substack{(P,Q)\in\mathcal{R}_{r-1-p}\\ P\subset P_\lambda}} H^{\mathrm{DR}}_{-q}(G \times^{P_\lambda \cap Q}(S_{P_\lambda \cap Q} \cap X_P)/X) \xrightarrow{d} \cdots \xrightarrow{\epsilon} H^{\mathrm{DR}}_{-q}(G \times^{P_\lambda} S_{P_\lambda}/X) \to 0$$

We claim that this subcomplex is exact. If $P \subset P_\lambda$ then $S_{P_\lambda \cap Q,\lambda} \subset X_P$ according to lemma 4.8.

Hence the complex (25) may be written as :

(26)

$$\cdots \xrightarrow{d} \oplus_{\substack{(P,Q)\in\mathcal{R}_{r-1-p}\\ P\subset P_\lambda}} H^{\mathrm{DR}}_{-q}(G \times^{P_\lambda \cap Q} S_{P_\lambda \cap Q}/X) \xrightarrow{d} \cdots \xrightarrow{\epsilon} H^{\mathrm{DR}}_{-q}(G \times^{P_\lambda} S_{P_\lambda}/X) \to 0$$

We may filter (26) according to $R = P_\lambda \cap Q$. As associated quotients we obtain complexes

$$(Z^\cdot \xrightarrow{\epsilon} \mathbb{Z}) \otimes H^{\mathrm{DR}}_{-q}(G \times^R S_{R,\lambda}/X)$$

if $R = P_\lambda$ and

$$Z^\cdot \otimes H^{\mathrm{DR}}_{-q}(G \times^R S_{R,\lambda}/X)$$

otherwise.

Here $Z^\cdot$ in degree $-p$ is $\mathbb{Z}\mathcal{U}_p^{(R)}$ where

$$\mathcal{U}_p^{(R)} = \{(P,Q) \in \mathcal{R}_{r-1-p} \mid P \subset P_\lambda, P_\lambda \cap Q = R\}$$

which may be simplified to

$$\mathcal{U}_p^{(R)} = \{(P,R) \in \mathcal{Q} \times \mathcal{Q} \mid R \subset P \subset P_\lambda\}$$

It is now easy to see that $Z^\cdot$ is isomorphic to a complex of the form (12). Hence it is acyclic, unless $R = P_\lambda$. In that case the complex $Z^\cdot \to \mathbb{Z}$ is the complex $0 \to \mathbb{Z} \xrightarrow{\pm 1} \mathbb{Z} \to 0$ and hence it is also exact.

To show that (25) is exact, it is sufficient that the complex below, which is the quotient of the complexes (24) and (25), is exact.

$$\cdots \xrightarrow{d} \oplus_{\substack{(P,Q)\in\mathcal{R}_{r-1-p} \\ P \not\subset P_\lambda}} H^{\mathrm{DR}}_{-q}(G \times^{P_\lambda \cap Q} (S_{P_\lambda\cap Q} \cap X_P)/X) \xrightarrow{d} \cdots \tag{27}$$

We now filter (27) according to $R = P_\lambda \cap Q$ and P. The associated quotient complexes are

$$Z'^\cdot \otimes H^{\mathrm{DR}}_{-q}(G \times^R (S_{R,\lambda} \cap X_P)/X)$$

$Z'^\cdot$ in degree $-p$ is $\mathbb{Z}\mathcal{U}_p'^{(P,R)}$, where

$$\mathcal{U}_p'^{(P,R)} = \{(P,Q) \in \mathcal{R}_{r-1-p} \mid Q \cap P_\lambda = R\}$$

There is a unique, maximal $R_\lambda \in \mathcal{Q}$, with the property that $R_\lambda \cap P_\lambda = R$. Hence

$$\mathcal{U}_p'^{(P,R)} = \{(P,Q) \in \mathcal{Q} \times \mathcal{Q} \mid R \subset Q \subset P \cap R_\lambda\}$$

Therefore $Z'^\cdot$ is isomorphic to the dual of a complex of the form (12). If $P \not\subset P_\lambda$ then it is easy to show that $R \neq R_\lambda \cap P$. We conclude that $Z'^\cdot$ is acyclic. □

Proof. **of Theorem 5.2.1** This is now standard. Assume $\mathcal{C} = \mathcal{B}$ and hence $T = X_B$. Also $T_{P,Q} = X_P$ if $Q \subset P$. Consider the double complex

$$\cdots \xrightarrow{d} \oplus_{(P,Q)\in\mathcal{R}_{r-1-p}} \pi^Q_{G*} E^\cdot_{P,Q} \xrightarrow{d} \oplus_{(P,Q)\in\mathcal{R}_{r-p}} \pi^Q_{G*} E^\cdot_{P,Q} \xrightarrow{d} \cdots$$

The homology of its columns is $H^{\mathrm{DR}}_{-q}(G \times^Q X_P/X)$ whilst it follows from lemma 5.2.2 that its total homology is $H^{\mathrm{DR}}_{p-q}(GX_B/X) = H^{-p+q}_{X_u}(X, \mathcal{O}_X)$. This implies the existence of the spectral sequence (16). □

6. The proofs

In this section we will give the proofs of Theorem 1.2 and Theorem 1.3. The proofs of these theorems consist of a series of lemmas. It should be stressed that the lemmas are certainly not the strongest possible. We also prove Prop. 1.4.

Throughout this section G, B, R, d, X, Φ, $\cdots$ will have the same meaning as in section 4. We also put $h = \dim R^G$. P, Q will be parabolic subgroups of G, containing B, such that $Q \subset P$.

The roots of B will be the negative roots and Φ^+ will be the set of positive roots.

Lemma 6.1. *Let $\beta \in \mathbb{N}$. Any G-representation occurring in $H^\beta_{X^u}(X, \mathcal{O}_X)$ occurs in some*

$$H^{\mathrm{DR}}_{-\gamma}(G \times^Q X_P/X)$$

where

$$\gamma \leq \beta + l(G/B) \tag{28}$$

Proof. This follows immediately from Theorem 5.2.1. □

Lemma 6.2. *Any G-representation occurring in $H^{\mathrm{DR}}_{-\gamma}(G \times^Q X_P/X)$ occurs in some*

$$H^\delta_{G\times^Q X_P}(G \times^Q X, \wedge^{\delta'}\Omega_{G\times^Q X/X})$$

where

$$\delta \leq \gamma + 2\dim G/B \tag{29}$$

Proof. From the spectral sequence for hyper cohomology

$$E^1_{pq} : H^q_{G\times^Q X_P}(G \times^Q X, \wedge^p\Omega_{G\times^Q X/X}) \Rightarrow H^{\mathrm{DR}}_{-p-q+2\mathrm{dim} G/Q}(G \times^Q X_P/X)$$

we deduce that

$$-\gamma = -\delta - \delta' + 2\dim G/Q \tag{30}$$

Furthermore $\wedge^{\delta'}\Omega_{G\times^Q X/X} = 0$ unless

$$0 \leq \delta' \leq \dim G/Q \tag{31}$$

Combining (30) and (31) yields

$$\delta \leq \gamma + 2\dim G/Q$$

which implies (29). □

Lemma 6.3. π^B_Q *induces an isomorphism*

$$H^\delta_{G\times^B X_P}(G \times^B X, \pi^{B*}_Q \wedge^{\delta'} \Omega_{G\times^Q X/X}) \cong H^\delta_{G\times^Q X_P}(G \times^Q X, \wedge^{\delta'}\Omega_{G\times^Q X/X}) \tag{32}$$

Proof. The fibers of $\pi_Q^B : G \times^B X \to G \times^Q X$ are isomorphic to Q/B. If H is a Levy subgroup of Q then $Q/B \cong H/B \cap H$. Hence it follows from Bott's theorem that $H^i(Q/B, \mathcal{O}_{Q/B}) = 0$ for $i > 0$. This implies that $R^i \pi^B_{Q*} \mathcal{O}_{G\times^B X} = 0$ for $i > 0$. (32) follows from this last fact in a standard way. □

Lemma 6.4. *Any G-representation occurring in*

$$H^{\delta}_{G\times^B X_P}(G \times^B X, \pi_Q^{B*} \wedge^{\delta'} \Omega_{G\times^Q X/X})$$

occurs in some

$$H^{\epsilon'}(G/B, \pi_Q^{B*} \wedge^{\delta'} \Omega_{G/Q} \otimes_{\mathcal{O}_{G/B}} H^{\epsilon}_{X_P}(X, \mathcal{O}_X)\tilde{})$$

where

$$\epsilon \le \delta \tag{33}$$

Proof. There is a composite functor spectral sequence

$$E^2_{pq} : H^p(\mathcal{H}^q_{G\times^B X_P}(G\times^B X, \pi_Q^{B*}\wedge^{\delta'}\Omega_{G\times^Q X/X})) \Rightarrow H^{p+q}_{G\times^B X_P}(G\times^B X, \pi_Q^{B*}\wedge^{\delta'}\Omega_{G\times^Q X/X})$$

Furthermore it is easy to see that

$$\begin{aligned} &\mathcal{H}^q_{G\times^B X_P}(G \times^B X, \pi_Q^{B*} \wedge^{\delta'} \Omega_{G\times^Q X/X}) \\ &\cong \mathcal{H}^q_{G\times^B X_P}(G \times^B X, \pi_Q^{B*} \wedge^{\delta'} \Omega_{G/Q} \otimes_{\mathcal{O}_{G/B}} \mathcal{O}_{G\times^B X}) \\ &\cong \pi_Q^{B*} \wedge^{\delta'} \Omega_{G/Q} \otimes_{\mathcal{O}_{G/B}} \mathcal{H}^q_{G\times^B X_P}(G \times^B X, \mathcal{O}_{G\times^B X}) \\ &\cong \pi_Q^{B*} \wedge^{\delta'} \Omega_{G/Q} \otimes_{\mathcal{O}_{G/B}} H^q_{X_P}(X, \mathcal{O}_X)\tilde{} \end{aligned}$$

It follows from these facts that

$$\delta = \epsilon + \epsilon'$$

But clearly $\epsilon' \ge 0$. □

In the sequel, let $\overline{\rho}$ be half the sum of the positive roots.

Lemma 6.5. *Any representation occurring in*

$$H^{\epsilon'}(G/B, \pi_Q^{B*} \wedge^{\delta'} \Omega_{G/Q} \otimes_{\mathcal{O}_{G/B}} H^{\epsilon}_{X_P}(X, \mathcal{O}_X)\tilde{}) \tag{34}$$

has a highest weight of the form

$$\chi_{\mathrm{hi}} = w(\chi_1 + \overline{\rho}) - \overline{\rho} \tag{35}$$

where $w \in \mathcal{W}_G$, $\chi_1 \in X(T)$ and χ_1 is of the form

$$\chi_1 = \sum_{\rho\in S} \rho + \chi_2$$

with $S \subset -\Phi^+$ and $\chi_2 \in X(T)$ a character, occurring in $H^{\epsilon}_{X_P}(X, \mathcal{O}_X)$.

Proof. Let $[e] \in G/B$ be the class of the unit element. Then, by Bott's theorem, any representation occurring in (34) must have a highest weight of the form (35) where χ_1 occurs in

$$(\pi_Q^{B*} \wedge^{\delta'} \Omega_{G/Q})_{[e]} \otimes_k H^{\epsilon}_{X_P}(X, \mathcal{O}_X)$$

Furthermore $(\pi_Q^{B*} \wedge^{\delta'} \Omega_{G/Q})_{[e]} = (\wedge^{\delta'} \mathbf{g}/\mathbf{q})^*$ where $\mathbf{g}$ and $\mathbf{q}$ are the Lie algebras of G and Q.

But any character occurring in $(\wedge^{\delta'} \mathbf{g}/\mathbf{q})^*$ will be of the form $\sum_{\rho\in S} \rho$ for some $S \subset -\Phi^+$. $\square$

Now we have to recall some facts and definitions from [9].

If $\lambda \in Y(T)_{\mathbb{R}}$ then we define $d_\lambda = \operatorname{codim} X_\lambda$, if λ, $\mu \in Y(T)_{\mathbb{R}}$ then $\lambda \sim \mu$ if $X_\lambda = X_\mu$. Finally, if $U \subset Y(T)_{\mathbb{R}}$ then $U_\lambda = \{\mu \in U \mid \mu \sim \lambda\}$

Lemma 6.6. *Assume that U is a closed bounded convex subset of $Y(T)_{\mathbb{R}}$, having dimension s. Then there is a T-equivariant filtration on $H^i_{X_U}(X, \mathcal{O}_X)$ such that*

$$\operatorname{gr} H^i_{X_U}(X, \mathcal{O}_X) = \oplus_{\lambda \in (U-\partial U)/\sim} \tilde{H}^{i+s-d_\lambda-1}(\Phi^{(U)}_\lambda) \otimes H^{d_\lambda}_{X_\lambda}(X, \mathcal{O}_X)$$

where

$$\Phi^{(U)}_\lambda = \overline{U_\lambda} - U_\lambda$$

Proof. The proof is identical to that of [9, Th. 3.4.1]. In loc. cit. this lemma was proved under the assumption that U is the unit ball, but this fact was not used. $\square$

Lemma 6.7. *Every T-character occurring in $H^{\epsilon}_{X_P}(X, \mathcal{O}_X)$ occurs in some $H^{d_\lambda}_{X_\lambda}(X, \mathcal{O}_X)$ where $\lambda \in Y(T)$. If we assume in addition that X has a T-stable point and that the character only occurs in $H^{d_\lambda}_{X_\lambda}(X, \mathcal{O}_X)$ when $\lambda = 0$ then*

$$\epsilon = d - \dim Z(G) \tag{36}$$

Proof. The first part of this lemma follows from applying lemma 6.6 to $U = \{\lambda \in Y(T)_{\mathbb{R}} \mid \|\lambda\| \leq 1\} \cap A_P$.

Assume that the hypothesis for the second part are fulfilled. First note that $\lambda = 0$ does not occur on the boundary of U if and only if $P = G$. In that case $\dim U = \dim A_G = \dim Z(G)$. Also $d_0 = d$ and if X has a stable point then $\Phi^{(U)}_0 = \emptyset$. We obtain that

$$\epsilon + \dim Z(G) - d - 1 = -1$$

which implies (36). $\square$

The following corollary summarizes the lemmas 6.1-6.7.

Corollary 6.8. *Assume that X has a stable point. Then every irreducible G-representation occurring $H^i_{X^u}(X, \mathcal{O}_X)$ has a highest weight of the form*

$$\chi_{\text{hi}} = \chi' + \sum_{\rho \in S'} \rho \tag{37}$$

with $S' \subset \Phi$ and χ' a T-character occurring in some $H^{d_\lambda}_{X_\lambda}(X, \mathcal{O}_X)$ where $\lambda \in Y(T)$.

If for every such decomposition of χ_{hi}, χ' only occurs in $H^{d_\lambda}_{X_\lambda}(X, \mathcal{O}_X)$ when $\lambda = 0$ then $i \geq h$.

Proof. It follows from lemmas 6.5, 6.7 that

$$\chi_{\text{hi}} = w(\sum_{\rho \in S} \rho + \chi_2 + \overline{\rho}) - \overline{\rho} \tag{38}$$

where $w \in \mathcal{W}_G$, $S \subset -\Phi^+$ and χ_2 occurs in some $H^{d_\lambda}_{X_\lambda}(X, \mathcal{O}_X)$ with $\lambda \in Y(T)$.

Put $\chi' = w\chi_2$. We may rewrite (38) as

$$\chi_{\text{hi}} = \chi' + w(\sum_{\rho \in S} \rho + \overline{\rho} - w^{-1}\overline{\rho})$$

It is clear that $\overline{\rho} - w^{-1}\overline{\rho}$ is a sum of positive roots. Since $S \subset -\Phi^+$, this proves (37).

If χ' only occurs in $H^{d_\lambda}_{X_\lambda}(X, \mathcal{O}_X)$ if $\lambda = 0$ then χ_2 has the same property. The second half of the lemma now follows by combining (28),(29),(33),(36) together with

$$\dim G = \dim Z(G) + 2 \dim G/B + l(G/B)$$

and the fact that if X has a stable point then

$$h = d - \dim G$$

$\square$

Proof. **of Theorem 1.2** Assume that R^G_χ is not Cohen-Macaulay. Then according to lemma 1.5 there must be a representation with character χ in $H^i_{X^u}(X, \mathcal{O}_X)$ where $i < h$. Then it follows from cor. 6.8 that there is a decomposition

$$\chi_{\text{hi}} = \chi' + \sum_{\rho \in S} \rho$$

where χ' occurs in some $H^{d_\lambda}_{X_\lambda}(X, \mathcal{O}_X)$ with $\lambda \neq 0$.

It follows from the hypothesis that χ' is good with respect to (T, W). On the other hand [9, cor. 3.3.2] implies that a good character cannot occur in $H^{d_\lambda}_{X_\lambda}(X, \mathcal{O}_X)$ where $\lambda \neq 0$. This is a contradiction. $\square$

Lemma 6.9. *Assume that X has a T-stable point and let χ be a T-character, strongly critical for (T, W). Then χ is good for (T, W).*

Proof. By definition there are $u_i \in]-1, 0]$ such that $\chi = \sum_i u_i \alpha_i$. Assume that the lemma is false, i.e. there exists some $\lambda \in Y(T) - \{0\}$ such that χ is not good for (T, W, λ). Then it is possible to write χ as

$$\chi = \sum_{i=1}^{d} a_i \alpha_i \tag{39}$$

where the $(a_i)_i$ are integers with the property that $a_i < 0$ if $i \in I_\lambda$ and $a_i \geq 0$ otherwise (recall that I_λ was defined by (2)).

It follows from the fact that X has a T-stable point that there exists an $i \in I_\lambda$ such that $\langle \lambda, \alpha_i \rangle > 0$.

(39) implies

$$\langle \lambda, \chi \rangle \leq -\sum_{i \in I_\lambda} \langle \lambda, \alpha_i \rangle$$

On the other hand

$$\langle \lambda, \chi \rangle = \sum_{i \in I_\lambda} u_i \langle \lambda, \alpha_i \rangle + \sum_{i \notin I_\lambda} u_i \langle \lambda, \alpha_i \rangle \tag{40}$$

$$> -\sum_{i \in I_\lambda} \langle \lambda, \alpha_i \rangle \tag{41}$$

which is a contradiction. $\square$

Proof. **of Theorem 1.3.1** This is now a direct consequence of Theorem 1.2 and lemma 6.9. $\square$

Lemma 6.10. *Assume that for any $\lambda \in Y(T) - \{0\}$, $|I_\lambda| \geq 2$. Let $\chi \in X(T)$ be critical for (T, W). Then χ is good for (T, W).*

Proof. Note that the hypotheses imply that X has a T-stable point.

Assume that the lemma is false. I.e. there is a $\lambda \in Y(T) - \{0\}$ together with integers $(a_i)_i$ such that

$$\chi = \sum_i a_i \alpha_i$$

where $a_i < 0$ iff $i \in I_\lambda$.

By definition it is possible to write $\chi = \sum_i u_i \alpha_i$, $u_i \leq 0$ in such a way that if there are integers $b_i \geq u_i$ with the property that $\chi = \sum b_i \alpha_i$ then $b_i \geq 0$.

Take $j \in I_\lambda$. Then it follows that for any integer b, $u_j \le b < 0$, $\chi - b\alpha_j$ is not in the semigroup generated by $(\alpha_i)_{i\neq j}$.

However, from the fact that $|I_\lambda| \ge 2$ for any $\lambda \neq 0$, it follows that the cone in $X(T)_\mathbb{R}$ spanned by $(\alpha_i)_{i\neq j}$ is $X(T)_\mathbb{R}$. It is then an easy exercise to show that the semigroup generated by $(\alpha_i)_{i\neq j}$ is actually a group.

This implies that $a_j < u_j$ since $\chi - a_j\alpha_j$ is in the group generated by $(\alpha_i)_{i\neq j}$.

Now we have (using the fact that there is an $i \in I_\lambda$ such that $\langle\lambda, \alpha_i\rangle > 0$)

$$\langle\lambda,\chi\rangle = \sum_{i\in I_\lambda} u_i\langle\lambda,\alpha_i\rangle + \sum_{i\notin I_\lambda} u_i\langle\lambda,\alpha_i\rangle \tag{42}$$

$$> \sum_{i\in I_\lambda} a_i\langle\lambda,\alpha_i\rangle + \sum_{i\notin I_\lambda} a_i\langle\lambda,\alpha_i\rangle \tag{43}$$

$$= \langle\lambda,\chi\rangle \tag{44}$$

which is a contradiction. □

Lemma 6.10 cannot be improved, as the following example shows :

Example 6.11. Let $G = T$ be a two dimensional torus. Then we identify $X(T)_\mathbb{R} \cong Y(T)_\mathbb{R} \cong \mathbb{R}^2$ in such a way that the pairing $\langle\ ,\ \rangle$ is given by the usual inner product on $\mathbb{R}^2$: $\langle(x_1,y_1),(x_2,y_2)\rangle = x_1x_2 + y_1y_2$.

Let $\alpha_1 = (-1,1)$, $\alpha_2 = (0,1)$, $\alpha_3 = (1,1)$, $\alpha_4 = (2,-3)$ and $\chi = -\alpha_1 - \alpha_2 = (1,-2)$. Then χ is not good for (T,W) (take $\lambda = (-2,1)$).

However
$$\chi = (-\frac{5}{4})\alpha_1 + (-\frac{1}{2})\alpha_2 + (-\frac{1}{4})\alpha_3$$
which easily shows that χ is critical for (T,W).

Proof. **of Theorem 1.3.2** Assume that the result is false. According to Theorem 1.2 and lemma 6.10 there must be a T-invariant linear subspace of codimension one in X having no T-stable point. Since X has a G-stable point this linear subspace must be G-invariant. This contradicts the hypothesis. □

Lemma 6.12. *Assume that for any $\lambda \in Y(T) - \{0\}$, $|I_\lambda| \ge 2$ and for every $j \in \{1,\dots,d\}$, α_j is in the group generated by $(\alpha_i)_{i\neq j}$.*

Let χ be in the group generated by $(\alpha_i)_i$. If χ is critical for (T,W) then χ is strongly critical for (T,W).

Proof. Assume that χ is critical and write $\chi = \sum u_i\alpha_i$, $u_i \le 0$ in such a way that if there are integers $b_i \ge u_i$ with the property that $\chi = \sum b_i\alpha_i$ then $b_i \ge 0$.

We may assume that there is some j such that $u_j \leq -1$ for otherwise the lemma is trivial.

It follows from the definition of critical that $\chi + \alpha_j$ is not in the semigroup generated by $(\alpha_i)_{i \neq j}$ and it follows as in lemma 6.10 that this semigroup is actually a group.

Furthermore, by hypothesis, it is possible to find integers $(c_i)_i$ such that $\chi = \sum c_i \alpha_i$. Hence $(1 + c_j)\alpha_j$ is not in the group generated by $(\alpha_i)_{i \neq j}$. This is a contradiction. $\square$

Proof. **of proposition 1.4** We want to apply lemma 6.12. From the fact that G is semisimple it follows that any G-invariant subspace of codimension one in X has a stable point. It is easy to see that this implies that for all $\lambda \in Y(T) - \{0\}$, $|I_\lambda| \geq 2$.

Furthermore it is well known, and easy to prove, that the only $\mathcal{W}_G$-invariant weight of a semisimple group is the trivial weight.

Let α_j be a weight of W. If $\alpha_j = 0$ then α_j is always in $\sum_{i \neq j} \mathbb{Z}\alpha_i$. Assume that $\alpha_j \neq 0$ and let $\mathcal{W}'$ be its stabilizer. Then

$$\sum_{\overline{w} \in \mathcal{W}_G/\mathcal{W}'} w\alpha_j$$

is $\mathcal{W}_G$-invariant and hence 0. Hence α_j is in $\sum_{i \neq j} \mathbb{Z}\alpha_i$.

Let $\chi | T = \sum_{i=1}^{l} \chi_i$. From the fact that $R^G \neq \{0\}$ it follows that $\chi_i \in \sum_i \mathbb{Z}\alpha_i$ and it is also a standard fact that $\Phi \subset \sum_i \mathbb{Z}\alpha_i$ if not all α_i are trivial (which is excluded by the fact that X has a stable point).

Hence by lemma 6.12 all $\chi_i - \sum_{\rho \in S} \rho$ $(i = 1, \ldots, l,\ S \subset \Phi)$ will be strongly critical for (T, W). This implies that χ is strongly critical for (G, W). $\square$

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