LOCAL COHOMOLOGY OF MODULES OF COVARIANTS

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ABSTRACT. Let G be a connected reductive algebraic group over an algebraically closed field of characteristic zero and let W, U two finite dimensional representations of G. In this paper we compute the local cohomology of $(U \otimes SW)^G$ provided a certain relatively weak technical condition is true.

CONTENTS

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1. Summary of notation

Symbol	**Meaning**	**Section**
G	a (split) connected reductive algebraic group	§2,§5.1
k	a basefield (usually of char. zero)	§3.2,§4.1
$\mathcal{D}_X$	the sheaf of differential operators on X	§3.1
$\mathcal{D}_X$-qch	the category of quasi-coherent $\mathcal{D}_X$-modules	§3.1
$(G, \mathcal{D}_X)$-qch	the category of G-equivariant quasi-coherent $\mathcal{D}_X$-modules	§3.1
$\mathcal{L}(Z, X)$	the holonomic module giving the intersection homology of Z	§3.1
$\Omega^{\cdot}_{Y/X}$	the relative De Rham complex of Y/X	App. A
$(\mathbb{Z}_l)_X$-mod	l-adic (not necessarily constructible) sheaves	§3.2
$D^b_c(X, \mathbb{Q}_l)$	the derived category of the constructible l-adic sheaves	§3.2
$\underline{\underline{\Gamma}}_U(\mathcal{F})$	the sheaf of sections of $\mathcal{F}$ with support in U	§3.2
$\underline{\underline{\Gamma}}_{U',U}(\mathcal{F})$	a certain map $\underline{\underline{\Gamma}}_U(\mathcal{F}) \to \underline{\underline{\Gamma}}_{U'}(\pi_*\mathcal{F})$	§3.2
$\mathrm{Tr}_\pi(\mathcal{F})$	the adjoint map $R\pi_!(R\pi^!\mathcal{F}) \to \mathcal{F}$ and various derived maps	§3.2
$\pi_!$, $\pi^!$, π_*, π^*	classical functors associated with an application π	§3.2
$C(\mathcal{C})$	complexes over $\mathcal{C}$	§4
$K(\mathcal{C})$	complexes over $\mathcal{C}$ with homotopy classes of maps	§4
$CF(\mathcal{C})$	filtered complexes over $\mathcal{C}$	§4
$KF(\mathcal{C})$	filtered complexes over $\mathcal{C}$ with homotopy classes of maps	§4
$C(\mathcal{T}, K(\mathcal{A}))$	a certain category	§4.1
$C(\mathcal{T}, \mathcal{A})$	a certain category	§4.2
$K(\mathcal{T}, K(\mathcal{A}))$	a certain category	§4.3
$K(\mathcal{T}, \mathcal{A})$	a certain category	§4.4
$C_0(\mathcal{T}, \mathcal{A})$	a certain category	§4.5
$K_0(\mathcal{T}, \mathcal{A})$	a certain category	§4.5
For	the forgetful functor $C(\mathcal{T}, \mathcal{A}) \to C(\mathcal{T}, K(\mathcal{A}))$	§4.5
Tot	the total complex of an object in $C(\mathcal{T}, \mathcal{A})$	§4.5
$\underline{\underline{\Gamma}}_Q$, $\underline{\underline{\Gamma}}_Y$	certain functors $C(\mathcal{T}, \mathcal{A}) \to C(\mathcal{T}, \mathcal{A})$	§4.6
T	a (split) maximal torus in G	§5.1
$\mathcal{W}_G$	the Weyl group of (G, T)	§5.1
Φ	the roots of (G, T)	§5.1
$X(T)$	the characters of T	§5.1
$Y(T)$	the one-parameter subgroups of T	§5.1
$\langle\ ,\ \rangle$	the natural pairing between $Y(T)$ and $X(T)$	§5.1
$(\ ,\ )$	a positive definite $\mathcal{W}_G$-invariant form on $Y(T)_\mathbb{R}$	§5.1
$\|\|\ \|\|$	the norm corresponding to $(\ ,\ )$ on $Y(T)_\mathbb{R}$	§5.1
B	a Borel subgroup of G containing T	§5.1
λ	usually an element of $Y(T)_\mathbb{R}$	
W	a finite dimensional representation of G	§5.1
d	$\dim W$	§5.1
$w_1, \dots, w_d$	a basis for W with diagonal T-action	§5.1
$\alpha_1, \dots, \alpha_d$	the weights corresponding to $w_1, \dots, w_d$	§5.1
R	the symmetric algebra of W over k	§5.1
X	the spectrum of R ($\cong W^*$)	§5.1
X_λ	a linear subspace of X associated to λ	§5.1
Y_λ	a linear subspace of X associated to λ	§5.1
P_λ	a parabolic subgroup associated to λ	§5.1

ρ	usually a root of G	
X_U	the union of all X_λ for $\lambda \in U$	§5.1
A_P	a polyhedral cone in $Y(T)_\mathbb{R}$ associated to P	§5.1
P, Q	usually parabolic subgroups of G	
X_P	the union of all X_λ for λ in A_P	§5.1
$S_{P,\lambda}$	locally closed subvarieties that form a stratification of PX_B	§5.1
$\mathcal{B}$	the indexing set for the stratification of PX_B	§5.1
$\mathcal{Q}$	the parabolic subgroups of G, containing B	§5.2
$l(P/Q)$	the length of the longest chain connecting Q to P in $\mathcal{Q}$	§5.2
r	the rank of the semi-simple part of G (equal to $l(G/B)$)	§5.2
$\mathcal{R}$	those (P,Q) in $\mathcal{Q} \times \mathcal{Q}$ with $P \supset Q$	§5.2
$\alpha_{Q,Q'}$	incidence numbers for the simplicial complex $\mathcal{Q}$	§5.2
$\alpha_{(P,Q),(P',Q')}$	incidence numbers for the simplicial complex $\mathcal{R}$	§5.2
$\beta_{(Q,Q)}$	an identification $H^{r-1}(\lvert\mathcal{R}\rvert, \mathbb{Z}) = \mathbb{Z}$	§5.2
C	the unit ball in $Y(T)_\mathbb{R}$	§5.2
C_Q	$A_Q \cap C$	§5.2
Ξ	$\{\alpha_1, \ldots, \alpha_d\} \cup \Phi$	§5.2
$\mathcal{P}$	a CW-complex on C associated to Ξ	§5.2
$\mathcal{P}_Q$	the CW-complex induced on C_Q by $\mathcal{P}$	§5.2
$\mathcal{P}_Q^\circ$	the interior of $\mathcal{P}_Q$	§5.2
$\alpha_{\sigma,\sigma'}$	incidence numbers for $\mathcal{P}$	§5.2
β_σ	an identification $H^{\dim C_Q}(C_Q, \partial C_Q, \mathbb{Z}) = \mathbb{Z}$	§5.2
$\alpha_{(\sigma,Q),(\sigma',Q')}$	numbers related to $\alpha_{(P,Q),(P',Q')}$	§5.2
$\pi_{Q,Q'}$	the projection $G \times^Q X \to G \times^{Q'} X$ or a related map	§5.3
${}^{\mathrm{perv}}\mathcal{H}$, ${}^{\mathrm{perv}}R$	perverse homology	§5.3
$\mathbf{F} \xrightarrow{g} \mathbf{E} \xrightarrow{f} \mathbf{D}$	certain objects in $C(X, \mathbb{Z}_l\text{-mod})$	§5.4
e_λ	$\operatorname{codim}(X_\lambda, X)$	§6
$\sim$	an equivalence relation on $Y(T)_\mathbb{R}$	§6
U_λ	those elements of U equivalent under $\sim$ to λ	§6
Λ	a special set of representatives for the quotient $C_B/\sim$	§6
Φ^+	the positive roots of G	§6
S	the simple roots of S	§6
H_λ	the Levy subgroup of P_λ associated to T	§6
$\mathcal{W}_\lambda$	the Weyl group of H_λ	§6
Φ_λ	the roots of H_λ	§6
Φ_λ^+	the positive roots of H_λ	§6
S_λ	the simple roots of H_λ	§6
$\mathcal{W}_{\lambda,G}$	a certain subset of $\mathcal{W}_\lambda$	§6
$A_B^{(w,\lambda)}$	a certain subset of A_B	§6
$E_{\lambda,Q}^{(Q)}$	a building block for the spectral sequence (5.7)	§6
$B_{\lambda,Q}^{(Q)}$	a combinatorial object	§6
σ	usually an element of $\mathcal{P}$	
w	usually an element of $\mathcal{W}_G$	
$l(w)$	the length of w in $\mathcal{W}_G$, with respect to S	§6
$P_{w,\lambda}$	a certain parabolic in G	§6
$\operatorname{relint} \sigma$	the relative interior of σ	§6
$\Psi_{w,\lambda}$	a certain closed subset of C_B	§6

f_λ	$\operatorname{codim}(GX_\lambda, X)$	§6
$\mathbb{Z}[M]$	the monoid ring over M	§7.3
$\mathbb{Z}\{M\}$	the infinite series over M	§7.3
$\mathbb{Z}[t][M]$	$\cong \mathbb{Z}[\mathbb{Z} \oplus M]$	§7.3
$\mathbb{Z}\{t\}\{M\}$	$\cong \mathbb{Z}\{\mathbb{Z} \oplus M\}$	§7.3
$\mathbf{P}$	a synonym for $X(T)_\mathbb{R}$	§7.3
$\mathbf{P}^{++}$	the dominant part of $\mathbf{P}$	§7.3
$V(\chi)$	the irreducible representation of G with highest weight χ	§7.3
$\mathcal{H}_T(V,t)$	the T equivariant Hilbert series of V	§7.3
$\mathcal{H}_G(V,t)$	the G equivariant Hilbert series of V	§7.3
$\overline{\rho}$	half the sum of the positive roots	§7.3
$\tilde{V}$	the vector bundle on G/Q-mod, associated to $V \in Q$-mod	§7.3
p	the projection $\mathbf{P} \to \mathbf{P}^{++}$ and various derived maps	§7.3
$X/\!\!/G$	the quotient of X by G	§7.3

2. Introduction

In this introduction and in part of this paper the base field will be $\mathbb{C}$. Let G be a connected reductive algebraic group and let W be a finite dimensional representation of G. Then G acts on the polynomial ring $R = SW$ and the Hochster-Roberts theorem [18] asserts that R^G is Cohen-Macaulay.

Now let U be an irreducible finite dimensional representation of G. It is well known that $(U \otimes_{\mathbb{C}} R)^G$ is not necessarily a Cohen-Macaulay R^G-module. Indeed, rather the opposite is true. Under rather weak conditions there are only a finite number of U such that $(U \otimes_{\mathbb{C}} R)^G$ is Cohen-Macaulay [7]. A conjecture that gives at least sufficient conditions for $(U \otimes R)^G$ to be Cohen-Macaulay was given in [24] by Stanley. A large part of this conjecture was proved in [28]. However already the torus case shows that the sufficient conditions given by this conjecture are usually not necessary.

Hence the problem we will try to attack in this paper is to give precise conditions for $(U \otimes R)^G$ to be Cohen-Macaulay. To be more precise, let $(R^G)^+$ be the positive part of R^G. We aim to calculate the local cohomology modules $H^i_{(R^G)^+}((U \otimes R)^G)$. Unfortunately the methods in this paper do not allow us to work in complete generality, and we will have to impose a condition on the action of G on W (condition (*) below). On the other hand we will show that this extra condition is relatively mild.

If $h = \dim R^G$ then it is well known that $(U \otimes R)^G$ is Cohen-Macaulay if and only if $H^i_{(R^G)^+}((U \otimes R)^G) = 0$, $i = 0, \dots, h-1$. It is also easy to see that $H^i_{(R^G)^+}((U \otimes R)^G) = (U \otimes H^i_I(R))^G$ where $I = \operatorname{rad} R(R^G)^+$ [27]. Hence one can compute $H^i_{(R^G)^+}((U \otimes R)^G)$ once one knows the G-structure of $H^i_I(R)$.

Let $X = \operatorname{Spec} R \cong W^*$. Then I is the defining ideal of the G-unstable locus X^u of X. I.e.

$$X^u = \{x \in X \mid o \in \overline{Gx}\}$$

and of course $H^i_I(R) = H^i_{X^u}(X, \mathcal{O}_X)$.

Let $\mathcal{D}_X$ be the sheaf of differential operators on $\mathcal{O}_X$. Then $\mathcal{H}^i_{X^u}(X, \mathcal{O}_X)$ carries a $\mathcal{D}_X$-module structure compatible with the G-action, and we propose to study the structure of $\mathcal{H}^i_{X^u}(X, \mathcal{O}_X)$ as quasi-coherent $(G, \mathcal{D}_X)$-module (see §3.1 for precise definitions).

Now let $T \subset G$ be a maximal torus and let $Y(T)$ be the abelian group of one-parameter subgroups of T. For $\lambda \in Y(T)$ define

$$X_\lambda = \{x \in X \mid \lim_{t \to 0} \lambda(t)x = 0\}$$
$$P_\lambda = \{g \in G \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists }\}$$

X_λ is a linear subspace of X and P_λ is a subgroup of X containing T and leaving X_λ stable. It is well known that P_λ is a parabolic subgroup of G [22, Prop. 2.6].

The Hilbert-Mumford criterion yields

$$X^u = \bigcup_{\lambda \in Y(T)} G X_\lambda$$

and there are natural projection maps

$$\pi_{P_\lambda, G} : G \times^{P_\lambda} X_\lambda \to G X_\lambda$$

The fact that P_λ is a parabolic subgroup of G implies that GX_λ is closed.

We now have introduced enough notation to state condition (*).

Condition (*).
1. If $\lambda, \lambda' \in Y(T)$ such that $X_\lambda \neq X_{\lambda'}$ then $GX_\lambda \neq GX_{\lambda'}$.
2. If $\lambda \in Y(T)$ then there exist $\lambda' \in Y(T)$ with $X_{\lambda'} = X_\lambda$ such that $\pi_{P_\lambda,G}$ is birational and small.

(A map $\pi : Y \to X$ is said to be small if for all $n > 0$, $\operatorname{codim}\{y \in Y \mid \dim \pi^{-1}y \geq n\} > 2n$).

Under condition (*), we can prove the following result.

Theorem 2.1. *Assume that condition (*) holds. Then $\mathcal{H}^n_{X^u}(X, \mathcal{O}_X)$, as an object of $(G, \mathcal{D}_X)$-qch has a finite filtration such that*

(2.1)
$$\operatorname{gr} \mathcal{H}^n_{X^u}(X, \mathcal{O}_X) = \oplus_{(w,\lambda)\ admissible} \tilde{H}^{n+\dim T - \operatorname{codim}(GX_\lambda, X) + l(w) - 1}(\Psi_{w,\lambda}, \mathbb{C}) \otimes \mathcal{L}(GX_\lambda, X)$$

Here the $\mathcal{L}(GX_\lambda, X)$ are simple holonomic G-equivariant $\mathcal{D}_X$-modules with regular singularities, whose De Rham complex is the intersection homology complex of GX_λ (suitably shifted).

$\tilde{H}^{n+\dim T - \operatorname{codim}(GX_\lambda, X) + l(w) - 1}(\Psi_{w,\lambda}, \mathbb{C})$ *is a finite dimensional vector space with trivial $(G, \mathcal{D}_X)$-structure.*

At this point there is a lot of unexplained notation in the statement of Theorem 2.1. These notations will be introduced in subsequent sections, but to help the reader we will give a summary at the end of this introduction. At this point we suffice by saying that the direct sum runs over a certain finite subset of the product of the Weyl group of G with $Y(T)_\mathbb{R}$.

To apply Theorem 2.1 one has to know the G-structure on $\mathcal{L}(GX_\lambda, X)$. This is the subject of Theorem 7.3.7 below where an explicit formula is given for the G-character of $\mathcal{L}(GX_\lambda, X)$ (under condition (*)).

How restrictive is condition (*) ? We will give two stable criteria for condition (*) to hold (Theorem 7.2.4 and Theorem 7.2.7 below). The first one says that (*) holds if the irreducible subrepresentations of W occur with high enough multiplicity. The second one, for simple groups, asserts that (*) holds if W has a simple subrepresentation having a big highest weight, lying in the root lattice.

A combination of these two results shows that if G is simple of adjoint type then (*) is satisfied for all but a finite number of W.

Our results contain of course the case when G is a torus since then (*) is always true. In particular Theorem 2.1 reduces to [29, Thm. 3.4.1].

In this paper we compute two more examples (see 7.4). If $G = \mathrm{Sl}(V)$, $\dim V = 2$, then (*) holds unless $W = V, S^2V$. Then we recover the results in [26] and [9] from Theorems 2.1 and 7.3.7.

If $G = \mathrm{Sl}(V)$, $\dim V = 3$, $W = V^m$ then (*) holds if $m \geq 3$. In that case we use Theorems 2.1 and 7.3.7 to determine when $(U \otimes R)^G$ is Cohen-Macaulay. It is shown that (if $m \geq 4$) there are exactly $(m-3)^2$ U's for which this is the case, whereas Stanley's criterion would only predict $\frac{(m-5)(m-4)}{2}$.

Now we give an outline of the proof of Theorem 2.1. In [28] a spectral sequence was constructed, using algebraic De Rham homology [16] which abuts to $H^i_{X^u}(X, \mathcal{O}_X)$. However the terms in this spectral sequence are of rather complicated nature, so it is difficult to draw conclusions.

A first observation is that this spectral sequence can be constructed in the more flexible framework of $\mathcal{D}$-modules and then we can use the methods of [29] for the torus case, to construct a more refined version with computable terms. However while it was clear that in the torus case the resulting spectral sequence was degenerate ([29, Thm. 3.4.1]) this is not at all clear for the general case.

Therefore use the Riemann-Hilbert correspondence to translate our problem to a problem about constructible sheaves. That is, we have to compute the perverse homology of $R\Gamma_{X^u}(X, \mathbb{C})$. Working in the framework of constructible sheaves has the added advantage that this formalism is more flexible since we are not restricted to smooth varieties.

Nevertheless it is still not clear why the resulting spectral sequence degenerates. It is conceivable that this would follow from some form of Hodge theory, but we have preferred to follow an alternative route (which is morally equivalent according to [11]). We work in the l-adic derived category. In that case there is an extra structure given by the Tate twists, and it turns out that the differentials in the E_2-term of the spectral sequence (5.7) are incompatible with it. Therefore they have to be zero.

Where does condition (*) come in ? Actually in two places. Firstly, we have to control somehow the perverse homology of $R\pi_{P_\lambda,G*}(\mathbb{Q}_{l\, G\times^{P_\lambda}X_\lambda})$ (or equivalently the homology of $(\pi_{P_\lambda,G}i)_+\mathcal{O}_{G\times^{P_\lambda}X_\lambda}$ where i is the inclusion $GX_\lambda \hookrightarrow X$). Condition (*) guarantees that this homology is a simple perverse sheaf (simple holonomic $\mathcal{D}_X$-module) whose support is GX_λ [15]. This puts a sharp constraint on the differentials in our spectral sequence.

Secondly, because the homology of $(\pi_{P_\lambda,G}i)_+\mathcal{O}_{G\times^{P_\lambda}X_\lambda}$ is in one degree, we can use Euler characteristics to compute its G-structure. This is the basis for the proof of Theorem 7.3.7.

Now we summarize the undefined notations in the statement of Theorem 2.1. Along the way we introduce some auxiliary notations which will come back in subsequent sections.

Let $X(T)$ be the character group of T and let $w_1, \dots, w_d$ be a basis of W for which the action of T is diagonal. Let $\alpha_1, \dots \alpha_d \in X(T)$ be the corresponding weights. It is easy to see that X_λ is a linear subspace of X, spanned by those w_i^* such that $\langle \lambda, \alpha_i \rangle < 0$ where $\langle\ ,\ \rangle$ is the natural pairing between $X(T)$ and $Y(T)$. P_λ is the subgroup of G containing T and having roots ρ such that $\langle \lambda, \rho \rangle \geq 0$. These descriptions still make sense for $\lambda \in Y(T)_\mathbb{R}$. Hence the notations X_λ, P_λ will also be used in this more general setting. It is still true that P_λ is parabolic and $P_\lambda X_\lambda = X_\lambda$.

Choose a Borel B containing T. The roots of B will be the negative roots. Φ, Φ^+, S will resp. be the roots, the positive roots and the simple roots of G. $\mathcal{W}_G$ will be the Weyl group of (G, T) and $l(w)$ will be the length of w in $\mathcal{W}_G$ with respect to S.

If $\lambda \in Y(T)_\mathbb{R}$ then H_λ is the Levy subgroup of P_λ and $\mathcal{W}_\lambda$, Φ_λ, Φ_λ^+ and S_λ are respectively the Weyl group of H_λ (i.e. the stabilizer of λ under the action of $\mathcal{W}_G$ on $Y(T)_\mathbb{R}$), the roots of H_λ (i.e. those roots such that $\langle \lambda, \rho \rangle = 0$), the positive roots of H_λ and the simple roots of H_λ.

If P is parabolic subgroup of G then we define $A_P = \{\lambda \in Y(T)_\mathbb{R} \mid P_\lambda \supset P\}$. If G is semi-simple then A_P is a simplicial cone in $Y(T)_\mathbb{R}$. We put $C_P = A_P \cap C$ where C is be the closed unit ball for a $\mathcal{W}_G$-invariant norm on $Y(T)_\mathbb{R}$.

If $\lambda, \lambda' \in Y(T)_\mathbb{R}$ then $\lambda \sim \lambda'$ if $X_\lambda = X_{\lambda'}$. This defines a $\mathcal{W}_G$-invariant equivalence relation on $Y(T)_\mathbb{R}$. If $U \subset Y(T)_\mathbb{R}$, $\lambda \in Y(T)_\mathbb{R}$ then $U_\lambda = \{\lambda' \in U \mid \lambda' \sim \lambda\}$.

We choose a set of representatives $\Lambda \subset C_B$ for the equivalence classes $C_B/\sim$ in such a way that if $\lambda \in \Lambda$ then $P_\lambda \supset P_{\lambda'}$ for all $\lambda' \sim \lambda$, $\lambda' \in C_B$. According to lemma 6.1, and the discussion thereafter, this is possible.

If $\lambda \in C_B$, $w \in \mathcal{W}_\lambda$ (i.e. $w\lambda = \lambda$) then

$$A_B^{(w,\lambda)} = A_B - \bigcup_{s \in S_\lambda \cap w^{-1}\Phi_\lambda^+} A_{P_s}$$

where P_s is the parabolic subgroup of G containing B and having s as a unique simple root.

A pair $(w, \lambda) \in \mathcal{W}_G \times \Lambda$ is called admissible if $w \in \mathcal{W}_\lambda$ and if $(A_B)_\lambda \cap A_B^{(w,\lambda)} \neq \emptyset$. For (w, λ) admissible one defines

$$\Psi_{w,\lambda} = \overline{(C_B \setminus \partial C)_\lambda \cap A_B^{(w,\lambda)}} - (C_B \setminus \partial C)_\lambda \cap A_B^{(w,\lambda)}$$

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3. Preliminaries

3.1. **G-equivariant $\mathcal{D}$-modules.** If X is is a scheme over $\mathbb{C}$, then we denote by $\mathcal{O}_X$-qch the category of quasi-coherent $\mathcal{O}_X$-modules.

We start with the following diagram of objects and maps

$$G \times G \times X \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{array} G \times X \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{s_0} \\ \xrightarrow{d_1} \end{array} X \tag{3.1}$$

$$\begin{aligned} d_0(g_1, x) &= g_1^{-1}x & \qquad d_0(g_1, g_2, x) &= (g_2, g_1^{-1}x) \\ d_1(g_1, x) &= x & d_1(g_1, g_2, x) &= (g_1g_2, x) \\ s_0(x) &= (e, x) & d_2(g_1, g_2, x) &= (g_1, x) \end{aligned}$$

Definition 3.1.1. [3] A G-equivariant quasi-coherent $\mathcal{O}_X$-module is a pair $(\mathcal{F}, \theta)$ where $\mathcal{F} \in \mathcal{O}_X$-qch and θ is an isomorphism $d_1^*\mathcal{F} \to d_0^*\mathcal{F}$ in $\mathcal{O}_{G\times X}$-qch such that

$$\begin{aligned} d_0^*\theta \circ d_2^*\theta &= d_1^*\theta \\ s_0^*\theta &= \mathrm{id}_{\mathcal{F}} \end{aligned} \tag{3.2}$$

The corresponding category is denoted by $(G, \mathcal{O}_X)$-qch.

If there is no possibility for confusion we will simply write $\mathcal{F}$ for $(\mathcal{F}, \theta)$.

Functors compatible with (flat or smooth) base-change preserve G-equivariance, since they preserve (3.2).

If X is a point then the category $(G, \mathcal{O}_X)$-qch is equivalent with the category or rational G-representations, that is vector spaces with a linear G-action, such that each vector is contained in a finite dimensional G-representation (as algebraic group).

Assume now that X is smooth. Let $\mathcal{D}_X$ be the sheaf of differential operators on X and denote by $\mathcal{D}_X$-qch the category of quasi-coherent $\mathcal{D}_X$ modules and by $D^b(\mathcal{D}_X\text{-qch})$ the associated derived category. We will identify $\mathcal{D}_X$-qch with its essential image in $D^b(\mathcal{D}_X\text{-qch})$.

If $\pi : X \to Y$ is a map between smooth schemes then π^* defines a functor $\mathcal{D}_Y$-qch $\to$ $\mathcal{D}_X$-qch and there is a formalism of direct and inverse images $\pi_!, \pi_+, \pi^!, \pi^+$ between appropriate subcategories of $D^b(\mathcal{D}_X\text{-qch})$ and $D^b(\mathcal{D}_Y\text{-qch})$ for which we refer the reader to [5].

Assume now that Y is a closed subset of X, that X is smooth and that X and Y are irreducible. An important object is $\mathcal{L}(Y, X)$ which is the holonomic $\mathcal{D}_X$-module whose De Rham complex is the intersection homology complex on Y (up to shift) [10]. $\mathcal{L}(Y, X)$ is the unique simple quasi-coherent submodule of $\mathcal{H}_Y^{\mathrm{codim}(Y,X)}(X, \mathcal{O}_X)$ whose support is Y.

A G-equivariant quasi-coherent $\mathcal{D}_X$-module is a pair $(\mathcal{F}, \theta)$ where $\mathcal{F}$ is in $\mathcal{D}_X$-qch and $\theta : d_1^*\mathcal{F} \to d_0^*\mathcal{F}$ is in $\mathcal{D}_{G\times X}$-qch such that (3.1) holds. Note that this makes sense since both $d_1^*\mathcal{F}$ and $d_0^*\mathcal{F}$ are in $\mathcal{D}_{G\times X}$-qch.

This implies that object in $(G, \mathcal{D}_X)$-qch are compatible with standard functors, since these commute with smooth base-change. We think in particular of π^*, $H^i\pi_+$, $H^i\pi^!$ for a G-equivariant map π and $\mathcal{H}_Y^i(X, -)$ for a G-equivariant closed subset Y of X. It also follows from the description above that $\mathcal{L}(Y, X)$ is in $(G, \mathcal{D}_X)$-qch.

Object in $(G, \mathcal{D}_X)$-qch are rather rigid, as is shown in the following proposition.

Proposition 3.1.2. *Assume that G is connected. Then the forgetful functor $(G, \mathcal{D}_X)$-qch $\to$ $\mathcal{D}_X$-qch is fully faithful. Furthermore if $\mathcal{M} \in (G, \mathcal{D}_X)$-qch and $\mathcal{N} \subset \mathcal{M}$ in $\mathcal{D}_X$-qch then $\mathcal{N} \in (G, \mathcal{D}_X)$-qch.*

Proof . This is presumably well-known, but I have not been able to locate a reference.

The proof below is a straightforward adaptation of the proof of [20, (1.9.1)]. It is based upon a generalization of [2, 4.2.5,4.2.6]. This generalization is deferred to the appendix.

Let $(\mathcal{M}, \phi)$, $(\mathcal{N}, \psi) \in (G, \mathcal{D}_X)$-qch and let $f \in \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})$. Then by A.1(1) there is a $g \in \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})$ such that $d_0^* g = \psi \circ d_1^* f \circ \phi^{-1}$. I.e. the following diagram is commutative.

$$\begin{tikzcd} d_1^*\mathcal{M} \arrow[r, "\psi"] & d_0^*\mathcal{N} \\ d_1^*\mathcal{M} \arrow[u, "d_1^* f"] \arrow[r, "\phi"] & d_0^*\mathcal{N} \arrow[u, "d_0^* g"'] \end{tikzcd}$$

Restricting to the unit section yields $f = g$. I.e. $f \in \operatorname{Hom}_{(G,\mathcal{D}_X)}(\mathcal{M}, \mathcal{N})$.

To show the second part of the Proposition, let $\mathcal{M} \in (G, \mathcal{D}_X)$-qch and $\mathcal{N} \subset \mathcal{M}$ in $\mathcal{D}_X$-qch. Then by A.1(2) there exist $\mathcal{N}' \subset \mathcal{M}$ in $\mathcal{D}_X$-qch such that $d_0^*\mathcal{N}' = \phi(d_1^*\mathcal{N})$. Restricting to the unit section yields $\mathcal{N}' = \mathcal{N}$. I.e. $\mathcal{N}$ is in $(G, \mathcal{D}_X)$-qch.

Remark 3.1.3. Actually we will use Proposition 3.1.2 only in the case of regular holonomic $\mathcal{D}$-modules.

Proposition 3.1.2 and the foregoing discussion dispense us to a certain extent of having to work with G-equivariant derived categories [3] (if G is connected).

That is, instead of directly computing in $(G, \mathcal{D}_X)$-qch we work in $\mathcal{D}_X$-qch (or in $D^b(\mathcal{D}_X\text{-qch})$) and in the end we know that we obtain G-equivariant sheaves, having a unique G-structure.

Sometimes G-equivariant quasi-coherent $\mathcal{D}_X$-modules are just too rigid. It is then useful to have the following weaker notion available [6].

A weakly G-equivariant quasi-coherent $\mathcal{D}_X$-module is a couple $(\mathcal{M}, \phi)$ with the usual properties, except that ϕ should only be in $\mathcal{O}_G \boxtimes \mathcal{D}_X$-qch. I.e. one only requires that ϕ is $\mathcal{O}_G$-linear, instead of $\mathcal{D}_G$-linear. The category of weakly G-equivariant quasi-coherent $\mathcal{D}_X$-modules is denoted by $(G, \mathcal{D}_X)$-wqch. Again these categories are stable under various natural functors.

The difference between $(G, \mathcal{D}_X)$-qch and $(G, \mathcal{D}_X)$-wqch maybe illustrated in the case that X is a point and G connected. In that case $(G, \mathcal{D}_X)$-qch is the category of $\mathbb{C}$-vector spaces, whereas $(G, \mathcal{D}_X)$-wqch is the category of rational G-representations.

3.2. **The l-adic derived category.** In this subsection X will be a scheme of finite type over a field k and l will be some integer, different from the characteristic of k.

In [12, §1.1.2] Deligne defined $D_c^b(X, \mathbb{Z}_l)$ (i.e. the derived category of l-adic sheaves with bounded constructible homology) as

$$2 - \operatorname{proj\,lim} D^b_{\mathrm{cftd}}(X, \mathbb{Z}/l^n) \tag{3.3}$$

where $D^b_{\mathrm{cftd}}(X, \mathbb{Z}/l^n)$ is the full subcategory of $D_c^b(X, \mathbb{Z}/l^n)$ consisting of complexes of finite Tor-dimension.

Deligne showed that if for any finite extension k' of k the groups $H^i(\mathrm{Gal}(\overline{k}/k'), \mathbb{Z}/l)$ are finite then (3.3) yields a triangulated category equipped with a t-structure whose heart is the category of constructible l-adic sheaves and hence one obtains a satisfactory theory. Furthermore the formalism of variance on $D^b_c(-, \mathbb{Z}_l)$ is directly derived from that of $D^b_c(-, \mathbb{Z}/l^n)$ by passage to the limit.

More recently people have given definitions of $D^b_c(X, \mathbb{Z}_l)$ which are valid in greater generality. For example in [2] a definition by O. Gabber is mentioned, which is unfortunately unpublished.

A published definition is that of Ekedahl [14] which we will follow in these notes. It constructs $D^b_c(X, \mathbb{Z}_l)$ as a triangulated subcategory of a derived category of an abelian category having enough injectives. This will be very useful for us.

We will now outline Ekedahl's construction in the least generality possible, and we will also change notations in such a way that they are more convenient for us in the sequel.

Denote by $(\mathbb{Z}_l)_X$-mod the ringed topos of inverse systems of sheaves on X : $\mathcal{F}_1 \leftarrow \mathcal{F}_2 \leftarrow \cdots \leftarrow \mathcal{F}_n \leftarrow \cdots$ where $\mathcal{F}_n$ is a sheaf of $(\mathbb{Z}/l^n)_X$-modules, and let $(\mathbb{Z}_l)_X$ itself stand for the object $(\mathbb{Z}/l)_X \leftarrow (\mathbb{Z}/l^2)_X \leftarrow \cdots$. ($(\mathbb{Z}_l)_X$-mod is not to be confused with the category of $\mathbb{Z}_l$-sheaves on X. This notion will never be used.)

Then $D^b_c(X, \mathbb{Z}_l)$ is a full triangulated subcategory of $D((\mathbb{Z}_l)_X\text{-mod})$ consisting of "normalized" complexes with bounded constructible homology (see loc. cit. for precise definitions).

If $\pi : X \to Y$ is a morphism of k-schemes of finite type then we have a pair of adjoint functors

$$\pi_* : (\mathbb{Z}_l)_X\text{-mod} \to (\mathbb{Z}_l)_Y\text{-mod}$$
$$\pi^* : (\mathbb{Z}_l)_Y\text{-mod} \to (\mathbb{Z}_l)_X\text{-mod}$$

which may be computed termwise on the inverse systems. π_* and π^* give rise to the corresponding functors on $D((\mathbb{Z}_l)_*\text{-mod})$ and $D^b_c(-, \mathbb{Z}_l)$.

There is another standard pair of adjoint functors

$$R\pi_! : D^b_c(X, \mathbb{Z}_l) \to D^b_c(Y, \mathbb{Z}_l)$$
$$R\pi^! : D^b_c(Y, \mathbb{Z}_l) \to D^b_c(X, \mathbb{Z}_l)$$

which is constructed in [14].

For maps

$$F \xrightarrow{i} X \xleftarrow{j} U$$

where i is a closed immersion and j is an open embedding there are functors $j_!, j_*, j^*, i^!\ i^*, i_*$ between the appropriate categories $(\mathbb{Z}_l)_F$-mod, $(\mathbb{Z}_l)_U$-mod and $(\mathbb{Z}_l)_X$-mod which also may be computed termwise. These functors satisfy the compatibilities of [2, §1.4] and hence a theory of perverse sheaves in $D^b_c(X, \mathbb{Z}_l)$ may be developed.

Suppose that $\pi : X \to Y$ is a proper surjective map between k-schemes of finite type. We say that π is a small resolution [15, §6.2] if π is birational, X is smooth and for all $n > 0$, $\operatorname{codim}\{y \in Y \mid \dim \pi^{-1}(y) \geq n\} > 2n$. We will make essential use of the following result.

Proposition 3.2.1. *Let $\pi : X \to Y$ be a small resolution and put $d = \dim Y$. Then $R\pi_*(\mathbb{Q}_l)_X[d]$ is a simple perverse sheaf on Y which gives rise to the intersection homology on Y, associated to the local system $(\mathbb{Q}_l)_Y$.*

Proof . This is a purely formal result once one has a theory of perverse sheaves. See loc. cit. for a proof in the topological case.

Now we introduce some supplementary notations which will be needed in the next sections.

If X is as above and $U \xrightarrow{i} X$ a locally closed embedding then for $\mathcal{F} \in (\mathbb{Z}_l)_X$-mod we denote $i_* i^! \mathcal{F}$ by $\underline{\underline{\Gamma}}_U(\mathcal{F})$. Since the left adjoint of the functor $\underline{\underline{\Gamma}}_U$ is $i_! i^*$ which is exact, $\underline{\underline{\Gamma}}_U$ preserves injectives. If i is a closed embedding then $\underline{\underline{\Gamma}}_U(\mathcal{F})$ is simply the sheaf of sections of $\mathcal{F}$ with support in $i(U)$.

If $Y \subset T \subset X$ are closed subsets of X such that $U = T - Y$ then there is an exact sequence

$$0 \to \underline{\underline{\Gamma}}_Y(\mathcal{F}) \to \underline{\underline{\Gamma}}_T(\mathcal{F}) \to \underline{\underline{\Gamma}}_U(\mathcal{F}) \tag{3.4}$$

in $(\mathbb{Z}_l)_X$-mod, which may be completed by 0 on the right if $\mathcal{F}$ is injective. Hence (3.4) gives rise to triangles in $D^+((\mathbb{Z}_l)_X\text{-mod})$ and $D^b_c(X, \mathbb{Z}_l)$.

If we have a morphism of k-schemes $\pi : X' \to X$ of finite type and closed subsets $T' \subset X'$, $T \subset X$ such that $\pi(T') \subset T$ then for $\mathcal{F} \in (\mathbb{Z}_l)_X$-mod there is obviously a unique map, which we will denote by $\underline{\underline{\Gamma}}_{T',T}(\mathcal{F})$, which makes the following diagram commutative

$$\begin{tikzcd} \pi_* \underline{\underline{\Gamma}}_{T'}(\mathcal{F}) \arrow[r] \arrow[d, "\underline{\underline{\Gamma}}_{T',T}(\mathcal{F})"] & \pi_* \mathcal{F} \arrow[d, "\mathrm{id}"] \\ \underline{\underline{\Gamma}}_T(\pi_* \mathcal{F}) \arrow[r] & \pi_* \mathcal{F} \end{tikzcd} \tag{3.5}$$

where the horizontal arrows are the natural injections. More generally if we have a diagram

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d] & X' \arrow[d, "\pi"] \\ Y \arrow[r] & T \arrow[r] & X \end{tikzcd} \tag{3.6}$$

where the horizontal maps are closed immersions and we put $U' = T' - Y'$, $U = T - Y$, then there is a map $\underline{\underline{\Gamma}}_{U',U}(\mathcal{F})$ which fits in the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi_* \underline{\underline{\Gamma}}_{Y'}(\mathcal{F}) \arrow[r] \arrow[d, "\underline{\underline{\Gamma}}_{Y',Y}(\mathcal{F})"] & \pi_* \underline{\underline{\Gamma}}_{T'}(\mathcal{F}) \arrow[r] \arrow[d, "\underline{\underline{\Gamma}}_{T',T}(\mathcal{F})"] & \pi_* \underline{\underline{\Gamma}}_{U'}(\mathcal{F}) \arrow[d, "\underline{\underline{\Gamma}}_{U',U}(\mathcal{F})"] \\ 0 \arrow[r] & \underline{\underline{\Gamma}}_Y(\pi_* \mathcal{F}) \arrow[r] & \underline{\underline{\Gamma}}_T(\pi_* \mathcal{F}) \arrow[r] & \underline{\underline{\Gamma}}_U(\pi_* \mathcal{F}) \end{tikzcd} \tag{3.7}$$

The definition uses the fact that $\pi^{-1}(U) \cap U'$ is open in U' and closed in $\pi^{-1}(U)$. $\underline{\underline{\Gamma}}_{U',U}(\mathcal{F})$ is the composition of the following maps $\pi_* \underline{\underline{\Gamma}}_{U'}(\mathcal{F}) = \pi_* \underline{\underline{\Gamma}}_{U'}(\mathcal{F}|X' - Y') \xrightarrow{\text{restr.}} \pi_* \underline{\underline{\Gamma}}_{\pi^{-1}(U)\cap U'}(\mathcal{F}|X' - \pi^{-1}(Y)) \xrightarrow{\underline{\underline{\Gamma}}_{\pi^{-1}(U)\cap U',U}} \underline{\underline{\Gamma}}_U(\pi_*(\mathcal{F})|X - Y) = \underline{\underline{\Gamma}}_U(\pi_* \mathcal{F})$.

From this description one deduces that $\underline{\underline{\Gamma}}_{U',U}(\mathcal{F})$ depends only on the data $(U', U, X', X, \pi, \mathcal{F})$ and not on the particular choice of Y, Y', T, T'.

If $\mathcal{F}$ is injective then the exact sequences in (3.7) may be completed with 0 on the right and hence they induce morphisms of triangles in $D^+((\mathbb{Z}_l)_X\text{-mod})$ and $D^b_c(X, \mathbb{Z}_l)$.

The following special case will be used in the following sections. Assume that $\pi : X' \to X$ is proper of finite type. Then the adjointness of the functors $R\pi_! = R\pi_*$

and $R\pi^!$ induce a trace map for $\mathcal{F} \in D^b_c(X, \mathbb{Z}_l)$, $\mathrm{Tr}_\pi(\mathcal{F}) : R\pi_*(R\pi^!\mathcal{F}) \to \mathcal{F}$. Usually we just write Tr_π if no confusion is possible. Now represent $R\pi^!\mathcal{F}$ and $\mathcal{F}$ by injective complexes $J^\cdot$ and $I^\cdot$ in $(\mathbb{Z}_l)_X$-mod and $(\mathbb{Z}_l)_{X'}$-mod respectively. Then Tr_π is represented by a map (determined up to homotopy) $\pi_* J^\cdot \to I^\cdot$, which we will also denote by Tr_π.

Now suppose that we have a diagram of the form (3.6). Then we will also denote by Tr_π the composition of maps $\pi_* \underline{\underline{\Gamma}}_{U'}(J^\cdot) \xrightarrow{\underline{\underline{\Gamma}}_{U',U}(J^\cdot)} \underline{\underline{\Gamma}}_U(\pi_* J^\cdot) \xrightarrow{\underline{\underline{\Gamma}}_U(\mathrm{Tr}_\pi)} \underline{\underline{\Gamma}}_U(I^\cdot)$. This map induces a map $\mathrm{Tr}_\pi : R\pi_* R\underline{\underline{\Gamma}}_{U'}(R\pi^!\mathcal{F}) \to R\underline{\underline{\Gamma}}_U(\mathcal{F})$ in $D^b_c(X, \mathbb{Z}_l)$, independent of the choices we have made.

We will need the following lemma, which gives a direct construction of Tr_π in the case of closed immersion. (That is using the right hand side of (3.8)).

Lemma 3.2.2. *Suppose that we have a diagram*

$$\begin{tikzcd} T' \arrow[r, "j"] \arrow[d, "\theta"] & X' \arrow[d, "\pi"] \\ T \arrow[r, "i"] & X \end{tikzcd}$$

where the horizontal maps are closed immersions, $\pi : X' \to X$ is proper, of finite type and $\mathcal{F} \in D^b(X, \mathbb{Z}_l)$. Then there is a commutative diagram

$$\begin{tikzcd} R\pi_* Rj_* Rj^! R\pi^! \mathcal{F} \arrow[r, "\cong"] \arrow[d, "Ri_*(\mathrm{Tr}_\theta(Ri^!\mathcal{F}))"] & R\pi_* R\underline{\underline{\Gamma}}_{T'}(R\pi^!\mathcal{F}) \arrow[d, "\mathrm{Tr}_\pi"] \\ Ri_* Ri^! \mathcal{F} \arrow[r, "\cong"] & R\underline{\underline{\Gamma}}_T(\mathcal{F}) \end{tikzcd} \tag{3.8}$$

Here the horizontal maps are the natural identifications. The left most vertical map is defined via the identification $R\pi_* Rj_* Rj^! R\pi^! \mathcal{F} = Ri_* R\theta_* R\theta^! Ri^! \mathcal{F}$.

Proof . Let $\mu : R\pi_* R\underline{\underline{\Gamma}}_{T'}(R\pi^!\mathcal{F}) \to R\underline{\underline{\Gamma}}_T(\mathcal{F})$ be the map which makes (3.8) commutative. Using the fact Tr is compatible with compositions of maps we can make the following commutative diagram

$$\begin{tikzcd} R\pi_* Rj_* Rj^! R\pi^! \arrow[r, "R\pi_*(\mathrm{Tr}_j(R\pi^!\mathcal{F}))"] \arrow[d, "Ri_*(\mathrm{Tr}_\pi(Ri^!\mathcal{F}))"] & R\pi_! R\pi^! \mathcal{F} \arrow[d, "\mathrm{Tr}_\pi(\mathcal{F})"] \\ Ri_* Ri^! \mathcal{F} \arrow[r, "\mathrm{Tr}_i(\mathcal{F})"] & \mathcal{F} \end{tikzcd} \tag{3.9}$$

Since $Rj_* Rj^! = R\underline{\underline{\Gamma}}_{T'}$, $Ri_* Ri^! = R\underline{\underline{\Gamma}}_T$ and since under these identifications, Tr_i, Tr_j are given by the natural transformations $R\underline{\underline{\Gamma}}_{T'} \to \mathrm{id}$, $R\underline{\underline{\Gamma}}_T \to \mathrm{id}$, combining (3.8) and (3.9) yields the following commutative diagram.

$$\begin{tikzcd} R\pi_* R\underline{\underline{\Gamma}}_{T'}(R\pi^!\mathcal{F}) \arrow[r] \arrow[d, "\mu"] & R\pi_* R\pi^! \mathcal{F} \arrow[d, "\mathrm{Tr}_\pi(\mathcal{F})"] \\ \underline{\underline{\Gamma}}_T \mathcal{F} \arrow[r] & \mathcal{F} \end{tikzcd} \tag{3.10}$$

Now $R\pi_* R\underline{\underline{\Gamma}}_{T'}(R\pi^!\mathcal{F})$ clearly has support in T and the standard triangle (3.4)

$$R\underline{\underline{\Gamma}}_T(\mathcal{F}) \to \mathcal{F} \to R\underline{\underline{\Gamma}}_{X-T}(\mathcal{F}) \to$$

where $\mathrm{Hom}^i(R\pi_* R\underline{\underline{\Gamma}}_{T'}(R\pi^! \mathcal{F}), R\underline{\underline{\Gamma}}_{X-T}(\mathcal{F})) = 0$ for $i \in \mathbb{Z}$, shows that μ is unique.

Since the definition of Tr_π shows that putting $\mu = \mathrm{Tr}_\pi$ makes (3.10) commutative. μ must be equal to Tr_π.

Remark 3.2.3. By replacing $R\pi_*$ by $R\pi_!$ the above statements remain valid for non-proper maps. This makes the discussion slightly more technical, and since it is not needed for the sequel, we have chosen not to include it.

4. On the construction of single complexes from double complexes when maps are only given up to homotopy

In this section we construct some machinery to deal with the technical problem that the trace map, discussed in the previous section, is only determined up to homotopy (unlike in the case of residual complexes, see [16]). Matters would be greatly simplified if there were a canonical way to define a trace map on l-adic shaves which is compatible with compositions of maps. Our approach here is a generalization of [23]).

In the sequel $\mathcal{A}_T$ will stand for $(\mathbb{Z}_l)_T$-mod and $\mathcal{A}$ will stand for the collection of categories $(\mathbb{Z}_l)_*$-mod. This is merely a hypothesis of convenience since for the most part, an arbitrary abelian category fibered over Sch may be used. Restricting to $(\mathbb{Z}_l)$-mod allows us to use without worries the formalism of inverse and direct images, outlined in Section 3.2.

If $\mathcal{C}$ is an abelian category then $C(\mathcal{C})$ is the category of complexes over $\mathcal{C}$ and $K(\mathcal{C})$ is the category of complexes modulo homotopy. $CF(\mathcal{C})$ and $KF(\mathcal{C})$ are the corresponding categories of filtered complexes.

We will consider structures of the form

$$\mathcal{T} = (P, e, S, (X_p)_{p\in P}, (\pi_{p,q})_{\substack{p,q\in P\\ p\le q}})$$

where

(1) P is a locally finite poset. I.e.

$$\forall p, q \in P : |\{r \in P \mid p \le r \le q\}| < \infty$$

(2) $e : P \to \mathbb{Z}$ is an order preserving map.
(3) S is a base scheme.
(4) The $(X_p)_p$ are S-schemes.
(5) $\pi_{p,q} : X_p \to X_q$ are S-morphisms such that $\pi_{p,p} = \mathrm{id}$ and $\pi_{q,r}\pi_{p,q} = \pi_{p,r}$.

A subset $Q \subset P$ is said to be catenary if for all $p, q \in Q$, $p \le q$ and all maximal chains $p = p_0 < p_1 < \cdots < p_n = q$ with $p_0, \cdots, p_n \in \mathcal{Q}$ have the same length and for such a maximal chain one has $n = e(q) - e(p)$.

We will use $\mathcal{T}$ to define several categories.

4.1. $C(\mathcal{T}, K(\mathcal{A}))$.

Objects : $((\mathcal{F}_p)_{p\in P}, (d_{p,q})_{\substack{p,q\in P\\ p\le q, e(q)=e(p)+1}})$ with

(1) $(\mathcal{F}_p)_p$ complexes in $C(\mathcal{A}_{X_p})$.
(2) $\{p \mid \mathcal{F}_p \neq 0\}$ is contained in a catenary subset of P.
(3) $d_{p,q} : \pi_{p,q*}\mathcal{F}_p \to \mathcal{F}_q$ maps of complexes with the property that for $p, q \in P$, $e(q) = e(p) + 2$

$$\sum_{\substack{p\le r\le q\\ e(r)=e(p)+1}} |d_{r,q}\pi_{r,q*}(d_{p,r})$$

is homotopic to zero.

Morphisms : If $\mathcal{F} = ((\mathcal{F}_p), (d_{p,q}))$, $\mathcal{G} = ((\mathcal{G}), (d_{p,q}))$ are in $C(\mathcal{T}, K(\mathcal{A}))$ then the elements of $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ are represented by $(f_{p,q})_{\substack{p,q\in P\\ p\le q, e(p)=e(q)}}$ where the $(f_{p,q})$

are maps of complexes $f_{p,q} : \pi_{p,q*}\mathcal{F}_p \to \mathcal{G}_q$ with the property that for $p, q \in P$, $e(q) = e(p) + 1$.

$$\sum_{\substack{p\le r\le q\\ e(r)=e(p)}} d_{r,q}\pi_{r,q*}(f_{p,r}) - \sum_{\substack{p\le r\le q\\ e(r)=e(q)}} f_{r,q}\pi_{r,q*}(d_{p,r}) \tag{4.1}$$

is homotopic to zero.

4.2. $C(\mathcal{T},\mathcal{A})$.

Objects : $((\mathcal{F}_p)_{p\in P}, (d_{p,q})_{\substack{p,q\in P\\ p\le q}})$ with

(1) $\mathcal{F}_p$ a $\mathbb{Z}$-graded object over $\mathcal{A}_{X_p}$. I.e. formally $\mathcal{F}_p = \oplus_{i\in\mathbb{Z}}\mathcal{F}_{p,i}$, $\mathcal{F}_{p,i} \in \mathcal{A}_{X_p}$.
(2) $\{p \mid \mathcal{F}_p \neq 0\}$ is contained in a catenary subset of P.
(3) $d_{p,q} : \pi_{p,q*}\mathcal{F}_p \to \mathcal{F}_q$ graded maps of degree $e(p)-e(q)+1$ with the property that for $p, q \in P$, $p \le q$

$$\sum_{p\le r\le q} d_{r,q}\pi_{r,q*}(d_{p,q}) = 0$$

Morphisms : If $\mathcal{F} = ((\mathcal{F}_p), (d_{p,q}))$, $\mathcal{G} = ((\mathcal{G}_p), (d_{p,q}))$ are in $C(\mathcal{T},\mathcal{A})$ then the elements of $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ are represented by maps $(f_{p,q})_{\substack{p,q\in P\\ p\le q}}$ where the $(f_{p,q})$ are graded maps $\pi_{p,q*}\mathcal{F}_p \to \mathcal{G}_q$ of degree $e(p)-e(q)$ with the property that for $p, q \in P$, $p \le q$

$$\sum_{p\le r\le q} d_{r,q}\pi_{r,q*}(f_{p,r}) - f_{r,q}\pi_{r,q*}(d_{p,r}) = 0$$

Homotopy : Let $\mathcal{F}$, $\mathcal{G}$ be as above and suppose that there are maps $f = (f_{p,q}) : \mathcal{F} \to \mathcal{G}$ in $C(\mathcal{T},\mathcal{A})$. Then a homotopy between f and g is represented by $(h_{p,q})_{p\le q}$ where the $h_{p,q} : \pi_{p,q*}\mathcal{F}_p \to \mathcal{G}_q$ are graded maps of degree $e(p) - e(q) - 1$ such that

$$f_{p,q} - g_{p,q} = \sum_{p\le r\le q} h_{r,q}\pi_{r,q*}(d_{p,r}) + d_{r,q}\pi_{r,q*}(h_{p,r})$$

4.3. $K(\mathcal{T},K(\mathcal{A}))$.

$K(\mathcal{T},K(\mathcal{A}))$ is defined as $C(\mathcal{T},K(\mathcal{A}))$ but now we suppose that the $d_{p,q}$ are homotopy classes and if $f = (f_{p,q})$ represents a morphism, then again the $f_{p,q}$ are homotopy classes.

4.4. $K(\mathcal{T},\mathcal{A})$.

$K(\mathcal{T},\mathcal{A})$ has the same objects as $C(\mathcal{T},\mathcal{A})$, but know $\mathrm{Hom}_{K(\mathcal{T},\mathcal{A})}(\mathcal{F},\mathcal{G})$ is equal to $\mathrm{Hom}_{C(\mathcal{T},\mathcal{A})}(\mathcal{F},\mathcal{G})$ modulo homotopy.

4.5. **Functors.** We will also define some functors between these categories. For : $C(\mathcal{T},\mathcal{A}) \to C(\mathcal{T},K(\mathcal{A}))$ (a forgetful functor, because it forgets some structure) sends an object $\mathcal{F} = ((\mathcal{F}_p), (d_{p,q})_{\substack{p,q\\ p\le q}}) \in C(\mathcal{T},\mathcal{A})$ to $\mathrm{For}(\mathcal{F}) = ((\mathcal{F}_p), (d_{p,q})_{\substack{p,q\\ e(q)=e(p)+1}})$ but now we consider $\mathcal{F}_p$ as a complex with differential $(-)^{e(p)}d_{p,p}$. It is easy to check that $\mathrm{For}(\mathcal{F})$ lies in $C(\mathcal{T},K(\mathcal{A}))$. The definition of For on maps is obvious.

Clearly, For factors to give a functor For : $K(\mathcal{T},\mathcal{A}) \to K(\mathcal{T},K(\mathcal{A}))$, also denoted by For.

Tot : $C(\mathcal{T},\mathcal{A}) \to CF(\mathcal{A})$ is the functor, which associates to an object in $C(\mathcal{T},\mathcal{A})$ its filtered total complex. Suppose that $\mathcal{F} = ((\mathcal{F}_p), (d_{p,q}))$ is in $C(\mathcal{T},\mathcal{A})$. Denote the structure map of $X_p \to S$ by π_p. Then $\mathrm{Tot}(\mathcal{F})$, as a graded object, is given by $\oplus_{p\in P}\pi_{p*}\mathcal{F}_p(-e(p))$ and the differential $\oplus\pi_{q*}(d_{p,q})$ makes it into a complex, i.e.

an object of $C(\mathcal{A}_S)$. Furthermore $\operatorname{Tot}(\mathcal{F})$ is equipped with an ascending filtration, defined as follows :

$$F_{-k}\operatorname{Tot}(\mathcal{F}) = \oplus_{e(p)\geq k}\pi_{p*}\mathcal{F}(-e(p)) \tag{4.2}$$

and $\operatorname{gr}\operatorname{Tot}(\mathcal{F})$ is given by $\oplus_{k\in\mathbb{Z}}\oplus_{e(p)=k}\pi_{p*}\mathcal{F}_p(-k)$ which leads to a spectral sequence for the homology of $\operatorname{Tot}(\mathcal{F})$. (Of course, at this stage, not necessarily convergent.)

$$E^1_{-uv} : \oplus_{e(p)=u}H^v(\pi_{p*}\mathcal{F}_p) \Rightarrow H^{v-u}(\operatorname{Tot}\mathcal{F}) \tag{4.3}$$

with differential $d : E^1_{-uv} \to E_{-u+1v}$ given by $\oplus_{\substack{e(p)=u,e(q)=u+1\\ p\leq q}}\pi_{q*}d_{p,q}$. An important fact is that the E^1-term of this spectral sequence only depends upon the image of $\operatorname{For}(\mathcal{F})$ in $K(\mathcal{T}, K(\mathcal{A}))$.

Tot obviously factors through a functor $K(\mathcal{T},\mathcal{A}) \to KF(\mathcal{A}_S)$ which we will denote by Tot too.

Example 4.5.1. Consider $\mathcal{T} = (\mathbb{Z}, \mathrm{id}, X, (X_p = X)_{p\in\mathbb{Z}}, (\pi_{p,q} = \mathrm{id})_{pq})$ Then $C(\mathcal{T}, K(\mathcal{A}))$ are complexes over $K(\mathcal{A})$, whereas the elements of $C(\mathcal{T},\mathcal{A})$ may be considered as double complexes with extra maps thrown in of degrees $(2,-1)$, $(3,-2)$, etc.... In case $\mathcal{A}$ is a module category, this situation has been studied in [23]

Now let $C_0(\mathcal{T}, K(\mathcal{A}))$ resp. $K_0(\mathcal{T}, K(\mathcal{A}))$ stand for the full subcategories of $C(\mathcal{T}, K(\mathcal{A}))$ and $K(\mathcal{T}, K(\mathcal{A}))$ whose objects $((\mathcal{F}_p)_p, (d_{p,q}))$ have the property that for all $p \leq q$, $\operatorname{Hom}^i_{K(\mathcal{A}_{X_q})}(\pi_{p,q*}\mathcal{F}_p, \mathcal{F}_q) = 0$ for $i < 0$.

Similarly we define $C_0(\mathcal{T},\mathcal{A})$ and $K_0(\mathcal{T},\mathcal{A})$ as the full subcategories of $C(\mathcal{T},\mathcal{A})$, $K(\mathcal{T},\mathcal{A})$ with objects $((\mathcal{F}_p), (d_{p,q}))$ such that for all $p \leq q$ and for all $i < 0$, $\operatorname{Hom}^i_{K(\mathcal{A}_{X_q})}(\pi_{p,q*}\mathcal{F}_p, \mathcal{F}_q) = 0$. Here $\mathcal{F}_p$ is made into a complex using differential $(-)^{e(p)}d_{p,p}$. Clearly $\operatorname{For}^{-1}(C_0(\mathcal{T}, K(\mathcal{A})) \subset C_0(\mathcal{T},\mathcal{A})$.

The following result is crucial for us.

Theorem 4.5.2. For *induces an equivalence between* $K_0(\mathcal{T},\mathcal{A})$ *and* $K_0(\mathcal{T}, K(\mathcal{A}))$.

Proof . The proof of this result is standard. See e.g. [23, Section 2] or [2, Prop 3.2.9] for similar results.

4.6. **Systems of support.** Let $\mathcal{T}$ be as before and let $Y = (Y_p)_{p\in P}$ a collection of closed subsets $Y_p \subset X_p$ with the property that $\pi_{p,q}(Y_p) \subset Y_q$ for $q \geq p$. Such an Y will be called a compatible system of supports.

If $\mathcal{F} = ((\mathcal{F}_p), (d_{p,q})) \in C(\mathcal{T},\mathcal{A})$ then we define

$$\underline{\underline{\Gamma}}_Y(\mathcal{F}) = (\underline{\underline{\Gamma}}_{Y_p}(\mathcal{F}_p), \underline{\underline{\Gamma}}_{Y_q}(d_{p,q}) \circ \underline{\underline{\Gamma}}_{Y_p,Y_q}(\mathcal{F}_p))$$

Let $T = (T_p)_p$, $Y = (Y_p)_p$ be compatible systems of supports where $T_p \subset Y_p$ for all $p \in P$. We put $U = Y - T \overset{\text{def}}{=} (Y_p - T_p)_{p\in P}$. Then we define

$$\underline{\underline{\Gamma}}_U(\mathcal{F}) = (\underline{\underline{\Gamma}}_{U_p}(\mathcal{F}_p), \underline{\underline{\Gamma}}_{U_q}(d_{p,q}) \circ \underline{\underline{\Gamma}}_{U_p,U_q}(\mathcal{F}_p))$$

and there is a "termwise" exact sequence

$$0 \to \underline{\underline{\Gamma}}_T(\mathcal{F}) \to \underline{\underline{\Gamma}}_Y(\mathcal{F}) \to \underline{\underline{\Gamma}}_U(\mathcal{F})$$

in $C(\mathcal{T},\mathcal{A})$ which may be completed by 0 on the righthand side if all $(\mathcal{F}_p)_p$ are injective in each degree.

Now suppose that $Q \subset P$ is a subset with the property that

$$\forall p, q \in Q, p \leq q, \forall r \in P, p \leq r \leq q \Rightarrow r \in Q \tag{4.4}$$

Define $\underline{\underline{\Gamma}}_Q(\mathcal{F}) = ((\mathcal{F}'_p), (d_{p,q}))$ where

$$\mathcal{F}'_p = \begin{cases} \mathcal{F}_p & \text{if } p \in Q \\ 0 & \text{otherwise} \end{cases}$$

and similarly

$$d'_{p,q} = \begin{cases} d_{p,q} & \text{if } p, q \in Q \\ 0 & \text{otherwise} \end{cases}$$

Then clearly $\underline{\underline{\Gamma}}_Q(\mathcal{F}) \in C(\mathcal{T}, \mathcal{A})$.

Now let $Q_1 \subset Q_2 \subset P$ be subsets with the property that $\forall p \in Q_i, \forall q \in P, q \geq p \Rightarrow q \in Q_i$. Then $Q = Q_2 - Q_1$ has property (4.4) and there is an exact sequence in $C(\mathcal{T}, \mathcal{A})$

$$0 \to \underline{\underline{\Gamma}}_{Q_1}(\mathcal{F}) \to \underline{\underline{\Gamma}}_{Q_2}(\mathcal{F}) \to \underline{\underline{\Gamma}}_Q(\mathcal{F}) \to 0$$

Note that $\underline{\underline{\Gamma}}_U$ and $\underline{\underline{\Gamma}}_Q$ may also be defined on maps, and hence are functors.

5. Some spectral sequences

5.1. **Stratifications.** This section summarizes the results in [28, Section 4], which give a generalization to the classical stratifications of the unstable locus of a representation. (See [17][19]). For the proofs we refer to loc. cit. They were stated for an algebraically closed base field but it is clear that they remain valid in the case we consider below.

Let k be a field of characteristic 0 and let G be a split connected reductive group over k. Let T, B be resp. a split maximal torus in G and a Borel subgroup containing T. Denote by Φ the set of roots of (G, T).

Let $X(T), Y(T)$ stand for the groups of characters and one-parameter subgroups of T. $\langle\ ,\ \rangle$ will be the natural pairing between $Y(T)$ and $X(T)$.

Let $\mathcal{W}_G$ be the Weyl group of (G, T). We will choose a positive definite, $\mathcal{W}_G$ invariant quadratic form $(\ ,\ )$ on $Y(T)_{\mathbb{R}}$. The corresponding norm will be denoted by $\|\ \|$. $Y(T)$ will be partially ordered by putting $\lambda < \lambda'$ if $\|\lambda\| < \|\lambda'\|$.

W will be a finite dimensional G-representation. We assume that W has a basis $w_1, \ldots, w_d$ for which the action of T is diagonal, with corresponding weights $\alpha_1, \ldots, \alpha_d \in X(T)$.

Let $R = SW$ and $X = \operatorname{Spec} R$. The closed points of X correspond to the elements of W^* and hence X is a linear space spanned by the dual basis $w_1^*, \ldots, w_d^*$, on which T acts with weights $-\alpha_1, \ldots, -\alpha_d$.

For $\lambda \in X(T)$ define

$$X_\lambda = \{x \in X \mid \lim_{t\to 0} \lambda(t)x = 0\}$$
$$P_\lambda = \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\}$$

Clearly $P_\lambda X_\lambda = X_\lambda$. Furthermore, it follows from [22, Proposition 2.5] that P_λ is a parabolic subgroup of G.

It is easy to see that X_λ is a linear subspace of X, spanned by those w_i^* such that $\langle\lambda, \alpha_i\rangle < 0$. P_λ is the subgroup of G containing T and having roots $\rho \in \Phi$ such that $\langle\lambda, \rho\rangle \geq 0$. These descriptions still make sense for $\lambda \in Y(T)_{\mathbb{R}}$. Hence the notations X_λ, P_λ will also be used in this more general setting. It is still true that P_λ is parabolic and $P_\lambda X_\lambda = X_\lambda$.

If $\lambda \in Y(T)_{\mathbb{R}}$ then we define Y_λ to be the linear subspace of X, spanned by those w_i^* such that $\langle\lambda, \alpha_i\rangle \leq -1$. By going to the Lie algebra, we see that $P_\lambda Y_\lambda = Y_\lambda$. Also $X_\lambda = Y_{n\lambda}$ for $n \gg 0$.

If $U \subset Y(T)_{\mathbb{R}}$ then we define $X_U = \bigcup_{\lambda\in U} X_\lambda$. If P is a parabolic subgroup of G, containing T then

$$A_P = \{\lambda \in Y(T)_{\mathbb{R}} \mid P_\lambda \supset P\}$$

I.e.

$$A_P = \{\lambda \in Y(T)_{\mathbb{R}} \mid \langle\lambda, \rho\rangle \geq 0 \text{ for all roots } \rho \text{ of } P\}$$

X_P will be defined as X_{A_P}. Using this notation, the Hilbert-Mumford criterion may be written as

$$X^u = GX_B$$

The parabolic subgroups of G, containing B form a combinatorial simplex and the A_P, as defined above, are a standard geometric realization of this simplex.

If $E \subset X$ then the set

$$\{\lambda \in A_B \mid E \subset Y_\lambda\} \tag{5.1}$$

is closed convex and hence, if it is non-empty, it contains a unique minimal element. We denote by $\mathcal{B}$ the set of elements of A_B that occur as minimal elements of set of the form (5.1). $\mathcal{B}$ is always a finite set. If $\lambda \in A_B$ and P is a parabolic subgroup of G, containing B, then PY_λ, PX_λ are closed in X. For $\lambda \in \mathcal{B}$ we define

$$S_{P,\lambda} = PY_\lambda - \cup_{\substack{\lambda'<\lambda \\ \lambda \in \mathcal{B}}} PY_{\lambda'}$$

Proposition 5.1.1. [28]

(1) *Let $\mathcal{C} \subset \mathcal{B}$ be a set with the property that $\lambda \in \mathcal{C}$, $\lambda' \in \mathcal{B}$, $\lambda' < \lambda$ implies $\lambda' \in \mathcal{C}$. Then*
$$\bigcup_{\lambda \in \mathcal{C}} S_{P,\lambda} = \bigcup_{\lambda \in \mathcal{C}} PY_\lambda$$
(2) $\bigcup_{\lambda \in \mathcal{B}} S_{P,\lambda} = PX_B$
(3) *If $\lambda, \lambda' \in \mathcal{B}$, $\lambda \neq \lambda'$ then $S_{P,\lambda} \cap S_{P,\lambda'} = \emptyset$.*
(4) $\overline{S}_{P,\lambda} \subset \bigcup_{||\lambda'|| \leq ||\lambda||} S_{P,\lambda'}$.
(5) *Let $\lambda \in \mathcal{B}$ and assume that P is a parabolic subgroup of G, containing B. Then $PS_{P_\lambda \cap P,\lambda} = S_{P,\lambda}$ and the natural map*
$$P \times^{P_\lambda \cap P} S_{P_\lambda \cap P,\lambda} \to S_{P,\lambda}$$
is settheoretically a bijection.

Remark 5.1.2. Using the methods of [28] it is easy to show that the map in 5.1.1(5) is actually an isomorphism. However we don't need this.

5.2. **Some complexes and their properties.** We keep the notations of section 5.1,

If $P \supset Q$ are parabolic subgroups of G, containing B and if there is a maximal chain
$$Q = P_0 \subset \cdots \subset P_u = P$$
then u will be denoted by $l(P/Q)$. We put $r = l(G/B)$, which is the rank of the semi-simple part of G. Define
$$\mathcal{Q} = \{\text{parabolic subgroups of } G, \text{ containing } B\}$$
If $Q, Q' \in \mathcal{Q}$ then we say that Q' is a face of Q if $Q \subset Q'$. Note that this is a change in convention with respect to [28]. The new convention is chosen in such a way that Q' is a face of Q if and only if $A_{Q'}$ is a face of A_Q.

The faces of dimension n in $\mathcal{Q}$ are given by
$$\mathcal{Q}_n = \{Q \in \mathcal{Q} \mid l(G/Q) = n + 1\}$$
Note that $\mathcal{Q}_{-1} = \{B\}$. Topologically B corresponds to the empty set which is by convention the boundary of every element of $\mathcal{Q}_0$.

The boundary maps $\partial : \mathbb{Z}\mathcal{Q}_n \to \mathbb{Z}\mathcal{Q}_{n-1}$ define incidence numbers $\alpha_{Q,Q'} \in \{\pm 1, 0\}$
$$\partial(Q) = \sum \alpha_{Q,Q'} Q'$$
We will also have occasion to use the following abstract complex
$$\mathcal{R} = \{(P, Q) \in \mathcal{Q} \times \mathcal{Q} \mid P \supset Q\}$$
where
$$\mathcal{R}_n = \{(P, Q) \in \mathcal{Q} \times \mathcal{Q} \mid l(P/Q) + n = r - 1\}$$

We let (P',Q') be a face of (P,Q) if $P' \supset P \supset Q \supset Q'$. This makes $\mathcal{R}$ into a abstract complex, whose corresponding topological space is an $r-1$-dimensional sphere. If we define $\alpha_{(P,Q),(P',Q')}$, $\beta_{(Q,Q)}$, as follows

$$\alpha_{(P,Q),(P',Q')} = \begin{cases} \alpha_{P,P'}(-)^{l(Q/B)} & \text{if } l(P'/P)=1,\ Q=Q' \\ \alpha_{Q',Q} & \text{if } l(Q/Q')=1,\ P=P' \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

$$\beta_{(Q,Q)} = (-)^{\lceil \frac{l(Q/B)}{2} \rceil} \tag{5.3}$$

then these define incidence numbers for $\mathcal{R}$, together with an identification of $H^{r-1}(|\mathcal{R}|,\mathbb{Z})$ with $\mathbb{Z}$ (the faces of maximal dimension in $\mathcal{R}$ are of the form (Q,Q), $Q \in \mathcal{Q}$).

Define $C = \{\lambda \in Y(T)_{\mathbb{R}} \mid \|\lambda\| \le 1\}$ and for $Q \in \mathcal{Q}$, $C_Q = C \cap A_Q$. We are going to define some particular CW-complex on C.

Let $F \subset Y(T)_{\mathbb{R}}$ be some convex polytope containing 0 in its interior and choose a homeomorphism $\phi : F \to C$ with the property that $\phi(0)=0$ and for all $p \in F$, $\phi(p)$ lies on the halfray starting in 0 and going through p.

Let $\Xi = \{\alpha_1,\dots,\alpha_d\} \cup \Phi$ (the reason for this particular choice of Ξ will become clear later). Then the hyperplanes in $Y(T)_{\mathbb{R}}$ defined by the elements of Ξ cut F up in pieces, and hence they define in a natural way the structure of a polyhedral complex on F. The image under ϕ of this polyhedral complex will be a regular CW-complex on C, which we will denote by $\mathcal{P}$ in the sequel (the elements of $\mathcal{P}$ will be the *closed* cells). By convention we consider the empty set as a cell in $\mathcal{P}$ of dimension -1 which is the boundary of every cell of dimension zero.

By our choice of ϕ, and the fact that $\Phi \subset \Xi$, for all $Q \in \mathcal{Q}$, C_Q will be a union of cells and hence $\mathcal{P}$ induces a CW-complex on C_Q, denoted by $\mathcal{P}_Q$.

We also define

$$\mathcal{P}_Q^\circ = \{\sigma \in \mathcal{P}_Q \mid \sigma \cap \operatorname{relint} C_Q \neq \emptyset\}$$

Proposition 5.2.1. *We may find $(\alpha_{\sigma,\sigma'})_{\sigma,\sigma' \in \mathcal{P}_B} \in \{\pm 1, 0\}$ and for all $Q \in \mathcal{Q}$, $(\beta_\sigma)_{\substack{\sigma \in \mathcal{P}_Q \\ \dim\sigma = \dim C_Q}}$ with the following properties*

1. *$\alpha_{\sigma,\sigma'} = 0$ unless σ' is a facet of σ. In that case $\alpha_{\sigma,\sigma'} \in \{\pm 1\}$.*
2. *If $\sigma, \sigma'' \in \mathcal{P}_B$ then*
$$\sum_{\sigma' \in \mathcal{P}_B} \alpha_{\sigma,\sigma'}\alpha_{\sigma',\sigma''} = 0$$
3. *$\beta_\sigma \in \{\pm 1\}$*
4. *Let $\sigma \in \mathcal{P}_Q^\circ$, $\dim\sigma = \dim C_Q - 1$. Then*
$$\beta_{\sigma_1'}\alpha_{\sigma_1',\sigma} + \beta_{\sigma_2'}\alpha_{\sigma_2',\sigma} = 0$$
where the σ_1', σ_2' are the two cells in $\mathcal{P}_Q$ having σ as a facet.
5. *Let $\sigma \in \mathcal{P}_Q$, $\dim\sigma = \dim A_Q$ and let $Q' \in \mathcal{Q}$, $Q' \subset Q$, $l(Q/Q') = 1$. Then there is a unique $\sigma' \in \mathcal{P}_{Q'}$ with the property that $\sigma \subset \partial\sigma'$. Furthermore*
$$\alpha_{\sigma',\sigma}\beta_{\sigma'} = \alpha_{Q',Q}\beta_\sigma$$
(Note that necessarily $\dim\sigma' = \dim A_{Q'}$)

Proof . We will content ourselves by giving the definition of the α's and the β's. The proof that they have the required properties is standard (similar to the verifications in [21, Chapter IV]).

Denote by C_Q^n the union of cells in $\mathcal{P}_Q$ of dimension less than or equal to n. To simplify the notation, we also define $C_Q^\vee = C_Q^{\dim C_Q - 1}$.

Let i_σ stand for the natural inclusions of pairs $(\sigma, \partial\sigma) \subset (C_Q^n, C_Q^{n-1})$ where $\dim \sigma = n$. Then the natural map

$$\oplus_{\substack{\sigma\in\mathcal{P}_Q\\ \dim\sigma=n}} H_n(\sigma,\partial\sigma) @>\oplus_\sigma i_{\sigma *}>> H_n(C_Q^n, C_Q^{n-1})$$

is an isomorphism [21, Theorem 2.1, Ch. IV]. Choose base vectors e_σ in $H_n(\sigma,\partial\sigma)$ for $\sigma \in \mathcal{P}_B$.

Then one defines $\partial i_{\sigma *}(e_\sigma) = \oplus_{\sigma'} \alpha_{\sigma,\sigma'} i_{\sigma' *}(e_{\sigma'})$ where ∂ is the natural boundary map

$$H_n(C_B^n, C_B^{n-1}) \to H_{n-1}(C_B^{n-1}, C_B^{n-2})$$

Now let $D = \{\lambda \in C_B \mid ||\lambda|| = 1\}$ and define $C^{(n)} = \bigcup_{\dim C_Q \le n} C_Q$. Let i_Q stand for the inclusion

$$(C_Q, \partial C_Q) \to (C^{(n)} \cup D, C^{(n-1)} \cup D)$$

where $n = \dim C_Q$.

$C^{(n)} \cup D$ is obtained from $C^{(n-1)} \cup D$ by attaching n-cells of the form C_Q. Hence

$$\bigoplus_{\substack{Q\in\mathcal{Q}\\ \dim C_Q = n}} H_n(C_Q, \partial C_Q) \xrightarrow{\oplus i_{Q*}} H_n(C^{(n)} \cup D, C^{(n-1)} \cup D)$$

is an isomorphism and one may choose base vectors $e_Q \in H_n(C_Q, \partial C_Q)$ such that

$$\partial i_{Q*}(e_Q) = \bigoplus_Q \alpha_{Q,Q'} i_{Q'*}(e_{Q'})$$

where ∂ is now the natural boundary map

$$H_n(C^{(n)} \cup D, C^{(n-1)} \cup D) \to H_{n-1}(C^{(n-1)} \cup D, C^{(n-2)} \cup D)$$

Having chosen the e_Q we define β_σ by

$$j_{Q*}(e_Q) = \oplus_\sigma \beta_\sigma i_{\sigma *}(e_\sigma)$$

where j_Q is the inclusion $(C_Q, \partial C_Q) \subset (C_Q, C_Q^\vee)$

Analogous with $\alpha_{(P,Q),(P',Q')}$ we define $\alpha_{(\sigma,Q),(\sigma',Q')}$ for $(\sigma, Q), (\sigma, Q')$ such that $\sigma' \subset \sigma$, $Q' \subset Q$, $\dim\sigma - \dim\sigma' + l(Q/Q') = 1$.

$$\alpha_{(\sigma,Q),(\sigma',Q')} = \begin{cases} \alpha_{\sigma,\sigma'}(-)^{l(Q/B)} & \text{if } \dim\sigma' - \dim\sigma = 1,\ Q = Q' \\ \alpha_{Q',Q} & \text{if } l(Q/Q') = 1,\ \sigma = \sigma' \\ 0 & \text{otherwise} \end{cases}$$

5.3. **The construction of the spectral sequences.** We keep the notations of the previous sections. In particular G, T, B, X, Φ, etc... will have their usual meaning.

Below we construct two spectral sequences abutting to ${}^{\mathrm{perv}}\mathcal{H}^n_{X^u}(X, \mathbb{Z}_l)$. Only the second one will be important to us afterwards. The first one is included because it is a direct generalization of [28, Thm 5.2.1], and also because it represents a resting point in the proof of the second one.

If $Q, Q' \in \mathcal{Q}$, $Q \subset Q'$ then $\pi_{Q,Q'}$ will be the projection map $G \times^Q X \to G \times^{Q'} X$. Clearly $\pi^*_{Q,Q'}(\mathbb{Z}_l)_{G\times^{Q'}X} = (\mathbb{Z}_l)_{G\times^Q X}$ and $R\pi^!_{Q,Q'}(\mathbb{Z}_l)_{G\times^{Q'}X} = (\mathbb{Z}_l)_{G\times^Q X}[2\dim Q'/Q](\dim Q'/Q)$.

The trace map (in $D^b_c(G \times^{Q'} X, \mathbb{Z}_l)$)

$$\mathrm{Tr}_{\pi_{Q,Q'}} : R\pi_{Q,Q'!} R\pi^!_{Q,Q'}(\mathbb{Z}_l)_{G\times^{Q'}X} = R\pi_{Q,Q'*}(\mathbb{Z}_l)_{G\times^Q X}[2\dim Q'/Q](\dim Q'/Q) \to (\mathbb{Z}_l)_{G\times^{Q'}X}$$

gives by twisting a map

(5.4)
$$R\pi_{Q,Q'*}(\mathbb{Z}_l)_{G\times^Q X}[2\dim G/Q](\dim G/Q) \to (\mathbb{Z}_l)_{G\times^{Q'} X}[2\dim G/Q'](\dim G/Q')$$

which we will denote by $\mathrm{Tr}_{\pi_{Q,Q'}}$ too.

As a convention we will denote other maps derived from $\mathrm{Tr}_{\pi_{Q,Q'}}$ by functoriality also by $\mathrm{Tr}_{\pi_{Q,Q'}}$. Noteworthy examples are maps induced on homology, perverse homology and the constructions in section 3.2.

Theorem 5.3.1. *There is a second quadrant spectral sequence, converging to ${}^{perv}\mathcal{H}^{-p+q}_{X^u}(X,\mathbb{Z}_l)$ with E^1-term*

$$E^1_{-pq} = \bigoplus_{(P,Q)\in\mathcal{R}_{r-1-p}} {}^{perv}R^{q+2\dim G/Q}(\pi_{Q,G*}\circ\underline{\underline{\Gamma}}_{G\times^Q X_P})(\mathbb{Z}_l)_{G\times^Q X}(\dim G/Q) \tag{5.5}$$

The differentials $d : E^1_{-pq} \to E_{-p+1q}$ are given by

$$\bigoplus \alpha_{(P',Q'),(P,Q)}\pi_{Q',G*}(\mathrm{Tr}_{\pi_{Q,Q'}}) \tag{5.6}$$

where the sum runs over all pairs

$$((P,Q),(P',Q')) \in \mathcal{R}_{r-1-p}\times\mathcal{R}_{r-p}$$

such that (P,Q) is a face of (P',Q').

Theorem 5.3.2. *There is a second quadrant spectral sequence, converging to ${}^{perv}\mathcal{H}^{-p+q}_{X^u}(X,\mathbb{Z}_l)$ with E^1 term*

(5.7)
$$E^1_{-pq} : \bigoplus_{\substack{\sigma\in\mathcal{P}_B, Q\in\mathcal{Q}, \sigma\subset C_Q\setminus\partial C\\ \dim\sigma-\dim T+l(Q/B)=-p}} {}^{perv}R^{q+2\dim G/Q}(\pi_{Q,G*}\circ\underline{\underline{\Gamma}}_{G\times^Q X_\sigma})(\mathbb{Z}_l)_{G\times^Q X}(\dim G/Q)$$

and the differentials $d : E^1_{-pq} \to E_{-p+1q}$ are given by

$$\bigoplus \alpha_{(\sigma',Q'),(\sigma,Q)}\pi_{Q',G*}(\mathrm{Tr}_{\pi_{Q,Q'}})$$

where the sum runs over all "permissible" pairs $((\sigma,Q),(\sigma',Q'))$. Permissible means that $\sigma\subset\sigma'$, $Q\subset Q'$, $\dim\sigma'-\dim\sigma+l(Q'/Q)=1$.

5.4. **Proofs of Theorems 5.3.1 and 5.3.2.** For $Q\in\mathcal{Q}$, let I_Q be an injective resolution of $(\mathbb{Z}_l)_{G\times^Q X}[2\dim G/Q](\dim G/Q)$ in $(\mathbb{Z}_l)_{G\times^Q X}$-mod.

The trace map defined by (5.4) gives rise to a map (determined up to homotopy)

$$\pi_{Q,Q'*}I_Q \to I_{Q'}$$

which we will also denote by $\mathrm{Tr}_{\pi_{Q,Q'}}$.

Lemma 5.4.1. $\mathrm{Hom}^i_{K((\mathbb{Z}_l)_{G\times^{Q'}X}\text{-}mod)}(\pi_{Q,Q'*}I_Q, I_{Q'}) = 0$ *for $i<0$.*

Proof . $R\pi_{Q,Q'*}$ has amplitude $[0,2\dim Q'/Q]$. Hence $\pi_{Q,Q'*}I_Q$ has homology in degrees $[-2\dim G/Q, -2\dim G/Q']$, whereas $I_{Q'}$ has homology only in degree $-2\dim G/Q'$. This proves the lemma.

We define now a poset $\mathcal{S}$ which, for technical reasons, is a union of three posets $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ respectively defined by

$$\begin{aligned}\mathcal{S}_1 &= \{(\sigma, Q) \mid Q \in \mathcal{Q}, \sigma \in \mathcal{P}_Q, \sigma \not\subset \partial C\}\\ \mathcal{S}_2 &= \{(C_P, Q) \mid P, Q \in \mathcal{Q}, P \supset Q\} \quad (\cong \mathcal{R})\\ \mathcal{S}_3 &= \{(C_B, G)\}\end{aligned}$$

$\mathcal{S}$ is ordered as follows

$$(U, Q) \le (U', Q') \iff Q \subset Q' \text{ and } U \subset U'$$

We define $e : \mathcal{S} \to \mathbb{Z}$ by

$$e(U, Q) = \begin{cases} \dim U + l(Q/B) - \dim T & \text{if } (U, Q) \in \mathcal{S}_1 \cup \mathcal{S}_2 \\ 0 & \text{if } (U, Q) \in \mathcal{S}_3 \end{cases}$$

It is easy to see that e is order preserving. Furthermore $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ are catenary subsets of $\mathcal{S}$.

For $(U, Q), (U', Q') \in \mathcal{S}$ we define $X_{(U,Q)}$ as $G \times^Q X$ and $\pi_{(U,Q),(U',Q')}$ as $\pi_{Q,Q'} : G \times^Q X \to G \times^{Q'} X$. It is easily seen that

$$\mathcal{T} = (\mathcal{S}, e, X, (X_{(U,Q)}), (\pi_{(U,Q),(U',Q')}))$$

satisfies the conditions listed in the beginning of section 4.

Now we proceed by constructing certain objects $\mathbf{D}, \mathbf{E}, \mathbf{F}$ in $C_0(\mathcal{T}, K(\mathbb{Z}_l\text{-mod}))$, related by maps $\mathbf{F} \xrightarrow{g} \mathbf{E} \xrightarrow{f} \mathbf{D}$. According to Theorem 4.5.2 these may be lifted (up to homotopy) to objects and maps

$$\tilde{\mathbf{F}} \xrightarrow{\tilde{g}} \tilde{\mathbf{E}} \xrightarrow{\tilde{f}} \tilde{\mathbf{D}} \tag{5.8}$$

in $C(\mathcal{T}, \mathbb{Z}_l\text{-mod})$. We will then construct a family of supports Y for $\mathcal{T}$ and we will show that $\mathrm{Tot} \circ \underline{\underline{\Gamma}}_Y$ applied to (5.8) yields quasi-isomorphims. The perverse homology of $(\mathrm{Tot} \circ \underline{\underline{\Gamma}}_Y)(\tilde{\mathbf{D}})$ will be ${}^{\mathrm{perv}}\mathcal{H}^*_{X^u}(X, \mathbb{Z}_l)$, whereas the spectral sequence (4.3) applied to $(\mathrm{Tot} \circ \underline{\underline{\Gamma}}_Y)(\tilde{\mathbf{E}})$ and $(\mathrm{Tot} \circ \underline{\underline{\Gamma}}_Y)(\tilde{\mathbf{F}})$ will yield the spectral sequences (5.5) and (5.7).

Now we proceed with the constructions. $\mathbf{D}$ will be $((\mathcal{F}_{(U,Q)}), (d_{(U,Q),(U',Q')}))$ where

$$\mathcal{F}_{(U,Q)} = \begin{cases} I_Q & \text{if } (U, Q) \in \mathcal{S}_3 \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathcal{S}_3$ is a singleton, $d_{(U,Q),(U',Q')}$ is always zero.

$\mathbf{E}$ will be $((\mathcal{F}_{(U,Q)}), (d_{(U,Q),(U',Q')}))$ where

$$\mathcal{F}_{(U,Q)} = \begin{cases} I_Q & \text{if } (U, Q) \in \mathcal{S}_2 \\ 0 & \text{otherwise} \end{cases}$$

and $d_{(U,Q),(U',Q')}$ will be zero unless $(U, Q) = (C_P, Q)$, $(U', Q') = (C'_P, Q')$ where (P, Q) is a facet of (P', Q'). In that case $d_{(U,Q),(U',Q')} = \alpha_{(P',Q'),(P,Q)} \operatorname{Tr}_{\pi_{Q,Q'}}$.

$\mathbf{F}$ will be $((\mathcal{F}_{(U,Q)}), (d_{(U,Q),(U',Q')})$ where

$$\mathcal{F}_{(U,Q)} = \begin{cases} I_Q & \text{if } (U, Q) \in \mathcal{S}_1 \\ 0 & \text{otherwise} \end{cases}$$

and $d_{(U,Q),(U',Q')}$ will be zero unless $(U,Q),(U',Q') \in \mathcal{S}_1$, $(U,Q) \leq (U',Q')$, $e(U',Q') = e(U,Q)+1$. In that case $d_{(U,Q),(U',Q')} = \alpha_{(U',Q'),(U,Q)} \operatorname{Tr}_{\pi_{Q,Q'}}$.

$f : \mathbf{E} \to \mathbf{D}$ will be a collection of maps $(f_{(U,Q),(U',Q')})$ where $f_{(U,Q),(U',Q')}$ is zero unless $(U,Q) = (C_Q,Q)$, $(U',Q') = (C_B,G)$. In that case $f_{(U,Q),(U',Q')}$ is equal to $\beta_{(Q,Q)} \operatorname{Tr}_{\pi_{Q,G}}$.

$g : \mathbf{F} \to \mathbf{E}$ will be a collection of maps $(g_{(U,Q),(U',Q')})$ where $g_{(U,Q),(U',Q')}$ is zero unless $(U,Q) = (\sigma,Q)$, $(U',Q') = (C_P,Q)$ where for some $(P,Q) \in \mathcal{R}$, $\sigma \in \mathcal{P}_P$, $\dim\sigma = \dim C_P$. In that case $g_{(U,Q),(U',Q')}$ is given by $\beta_U \mathrm{Id}_{I_Q}$.

One may verify, using Proposition 5.2.1 and lemma 5.4.1 that $\mathbf{D},\mathbf{E},\mathbf{F},f,g$ lie indeed in $C_0(\mathcal{T},K(\mathbb{Z}_l\text{-mod}))$.

As already said above, $\tilde{\mathbf{D}}$, $\tilde{\mathbf{E}}$, $\tilde{\mathbf{F}}$, $\tilde{f}$, $\tilde{g}$ will be liftings of $\mathbf{D},\mathbf{E},\mathbf{F},f,g$ to $C(\mathcal{T},\mathbb{Z}_l\text{-mod})$ under the functor For.

For $(U,Q) \in \mathcal{S}$ define $Y_{(U,Q)} = G \times^Q QX_U$. Clearly $\pi_{(U,Q),(U',Q')}(Y_{(U,Q)}) \subset Y_{(U',Q')}$ if $(U,Q) \leq (U',Q')$ in $\mathcal{S}$ and hence $Y = (Y_{(U,Q)})$ is a $\mathcal{T}$-compatible system of supports in the sense of section 4.6. Furthermore,

$$Y_{(U,Q)} = \begin{cases} G \times^Q X_U & \text{if } (U,Q) \in \mathcal{S}_1 \cup \mathcal{S}_2 \\ X^u & \text{if } (U,Q) \in \mathcal{S}_3 \end{cases}$$

Claim 1. $(\mathrm{Tot} \circ \underline{\underline{\Gamma}}_Y)(\tilde{f})$ is a quasi-isomorphism.

Proof . The proof is very similar to the proof of [28, Theorem 5.2.1]

Let $\mathcal{C}$ be a subset of $\mathcal{B}$ as in Proposition 5.1.1 and define

$$T_{\mathcal{C}} = \bigcup_{\lambda \in \mathcal{C}} S_{B,\lambda} = \bigcup_{\lambda \in \mathcal{C}} Y_\lambda$$

For $(U,Q) \in \mathcal{S}$ put $T_{\mathcal{C},U,Q} = QT_{\mathcal{C}} \cap QX_U$. Then $T_{\mathcal{C},U,Q}$ is a closed subset of X and $Y_{\mathcal{C}} = (G \times^Q T_{\mathcal{C},U,Q})_{(U,Q)\in\mathcal{S}}$ forms a $\mathcal{T}$-compatible family of supports.

Our aim is now to show, by induction on $|\mathcal{C}|$ that $(\mathrm{Tot} \circ \underline{\underline{\Gamma}}_{Y_{\mathcal{C}}})(\tilde{f})$ is a quasi-isomorphism. Obviously, the case we need is $\mathcal{C} = \mathcal{B}$ and the case $\mathcal{C} = \emptyset$ is trivial.

To start the induction let λ be a maximal element of $\mathcal{C}$ and put $\mathcal{C}' = \mathcal{C} - \lambda$. It follows form Proposition 5.1.1 that

$$QT_{\mathcal{C}} = \bigcup_{\lambda \in \mathcal{C}} QY_\lambda = \bigcup_{\lambda \in \mathcal{C}} S_{Q,\lambda}$$

Hence $T_{\mathcal{C},U,Q}$ is the disjoint union of $T_{\mathcal{C}',U,Q}$ and $S_{Q,\lambda} \cap QX_U$. Then $Y_\lambda \overset{\text{def}}{=} Y_{\mathcal{C}} - Y_{\mathcal{C}'} = (G \times^Q (T_{\mathcal{C},U,Q} - T_{\mathcal{C}',U,Q}))_{(U,Q)} = (G \times^Q (S_{Q,\lambda} \cap QX_U))_{(U,Q)}$.

From the discussions in section 4.6 we obtain a commutative diagram in $C(\mathcal{T},\mathbb{Z}_l\text{-mod})$ with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \underline{\underline{\Gamma}}_{Y_{\mathcal{C}'}}(\tilde{\mathbf{E}}) \arrow[r] \arrow[d, "\underline{\underline{\Gamma}}_{Y_{\mathcal{C}'}}(\tilde{f})"] & \underline{\underline{\Gamma}}_{Y_{\mathcal{C}}}(\tilde{\mathbf{E}}) \arrow[r] \arrow[d, "\underline{\underline{\Gamma}}_{Y_{\mathcal{C}}}(\tilde{f})"] & \underline{\underline{\Gamma}}_{Y_\lambda}(\tilde{\mathbf{E}}) \arrow[r] \arrow[d, "\underline{\underline{\Gamma}}_{Y_\lambda}(\tilde{f})"] & 0 \\
0 \arrow[r] & \underline{\underline{\Gamma}}_{Y_{\mathcal{C}'}}(\tilde{\mathbf{D}}) \arrow[r] & \underline{\underline{\Gamma}}_{Y_{\mathcal{C}}}(\tilde{\mathbf{D}}) \arrow[r] & \underline{\underline{\Gamma}}_{Y_\lambda}(\tilde{\mathbf{D}}) \arrow[r] & 0
\end{tikzcd}$$

Hence, by induction, it is now sufficient to show that $(\mathrm{Tot} \circ \underline{\underline{\Gamma}}_{Y_\lambda})(\tilde{f})$ is a quasi-isomorphism. To this end it is sufficient to show that this map induces an isomorphism between the E^2 terms of the spectral sequences associated to the natural filtrations (4.2) on $(\mathrm{Tot} \circ \underline{\underline{\Gamma}}_{Y_\lambda})(\tilde{\mathbf{D}})$ and $(\mathrm{Tot} \circ \underline{\underline{\Gamma}}_{Y_\lambda})(\tilde{\mathbf{E}})$.

The E_1-term of the spectral sequence for $(\mathrm{Tot}\circ \underline{\underline{\Gamma}}_{Y_\lambda})(\tilde{\mathbf{E}})$ looks like

$$E^1_{-pq}(\mathbf{E}) : \oplus_{(P,Q)\in\mathcal{R}_{-p-1+r}} R^{q+2\dim G/Q}(\pi_{Q,G*}\circ \underline{\underline{\Gamma}}_{G\times^Q(S_{Q,\lambda}\cap X_P)})(\mathbb{Z}_l)_{G\times^Q X}(\dim G/Q)$$

with differential similar to (5.6).

Similarly for $(\mathrm{Tot}\circ \underline{\underline{\Gamma}}_{Y_\lambda})(\tilde{D})$ we have

$$E^1_{-pq}(\mathbf{D}) = \begin{cases} R^q\,\underline{\underline{\Gamma}}_{S_{G,\lambda}}(X,\mathbb{Z}_l) & \text{if } p=0 \\ 0 & \text{otherwise} \end{cases}$$

The maps induced on these E^1-terms is zero everywhere, except in degrees $p=0$ where it is a map

$$\epsilon : E^1_{0q}(\mathbf{E}) \to E^1_{0q}(\mathbf{D}) : \oplus_{(P,Q)\in\mathcal{R}_{r-1}}\beta_{(Q,Q)}\,\mathrm{Tr}_{\pi_{Q,G}}$$

(note that $(P,Q)\in\mathcal{R}_{r-1} \iff P=Q$).

Hence to show that $\underline{\underline{\Gamma}}_{Y_\lambda}(\tilde{f})$ induces an isomorphism on E^2, we have to show that the following complexes, for varying q, are exact.

(5.9)

$$\begin{aligned} \cdots &\xrightarrow{d} \oplus_{(P,Q)\in\mathcal{R}_{r-1-p}} R^{q+2\dim G/Q}(\pi_{Q,G*}\circ \underline{\underline{\Gamma}}_{G\times^Q(S_{Q,\lambda}\cup X_P)})(\mathbb{Z}_l)_{G\times^Q X}(\dim G/Q) \\ &\xrightarrow{d} \oplus_{(P,Q)\in\mathcal{R}_{r-p}} R^{q+2\dim G/Q}(\pi_{Q,G*}\circ \underline{\underline{\Gamma}}_{G\times^Q(S_{Q,\lambda}\cup X_P)})(\mathbb{Z}_l)_{G\times^Q X}(\dim G/Q) \xrightarrow{d} \cdots \\ &\xrightarrow{\epsilon} R^q\,\underline{\underline{\Gamma}}_{S_{G,\lambda}}(X,\mathbb{Z}_l) \end{aligned}$$

This complex is similar to [28, eq. (21)] which was for algebraic De Rham homology. The proof now proceeds as in loc. cit.

Claim 2. $(\mathrm{Tot}\circ \underline{\underline{\Gamma}}_{Y})(\tilde{g})$ is a quasi-isomorphism.

Proof . We follow a method similar to the proof of Claim 1.

This time let $\mathcal{C}$ be a subset of $\mathcal{R}$ with the property that if $(P,Q)\in\mathcal{C}$ then all $(P',Q')\in\mathcal{R}$ such that $(P',Q')\geq(P,Q)$ are in $\mathcal{C}$. Put

$$\mathcal{S}_{\mathcal{C}} = \{(U,Q)\in\mathcal{S} \mid \forall P\in\mathcal{Q} : U\subset C_P \Rightarrow (P,Q)\in\mathcal{C}\}\cup\{(C_B,G)\}$$

Clearly if $(U,Q)\in\mathcal{S}_{\mathcal{C}}$ and $(U',Q')\in\mathcal{S}$ with $(U',Q')\geq(U,Q)$ then $(U',Q')\in\mathcal{S}_{\mathcal{C}}$. Obviously $\mathcal{S}_{\mathcal{R}}=\mathcal{S}$. Our aim is to show by induction on $|\mathcal{C}|$ that $(\mathrm{Tot}\circ \underline{\underline{\Gamma}}_{Y}\circ \underline{\underline{\Gamma}}_{\mathcal{S}_{\mathcal{C}}})(\tilde{g})$ is a quasi-isomorphism (using the notations of §3.6). This shows what we want since $\mathrm{Tot}\circ \underline{\underline{\Gamma}}_{Y}\circ \underline{\underline{\Gamma}}_{\mathcal{S}_{\mathcal{R}}} = \mathrm{Tot}\circ \underline{\underline{\Gamma}}_{Y}\circ \underline{\underline{\Gamma}}_{\mathcal{S}} = \mathrm{Tot}\circ \underline{\underline{\Gamma}}_{Y}$.

Let (P_0,Q_0) be a minimal element of $\mathcal{C}$ and put $\mathcal{C}'=\mathcal{C}-(P_0,Q_0)$.

Put also

$$\mathcal{S}_{(P_0,Q_0)} = \{(U,Q_0)\in\mathcal{S} \mid P_0 \text{ is the maximal element of } \mathcal{Q} \text{ such that } U\subset C_{P_0}\}$$

There are exact sequences in $C(\mathcal{T},\mathbb{Z}_l\text{-mod})$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\underline{\Gamma}}_{\mathcal{S}_{\mathcal{C}'}}(\tilde{\mathbf{F}}) & \longrightarrow & \underline{\underline{\Gamma}}_{\mathcal{S}_{\mathcal{C}}}(\tilde{\mathbf{F}}) & \longrightarrow & \underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{\mathbf{F}}) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \underline{\underline{\Gamma}}_{\mathcal{S}_{\mathcal{C}'}}(\tilde{g})} & & \Big\downarrow{\scriptstyle \underline{\underline{\Gamma}}_{\mathcal{S}_{\mathcal{C}}}(\tilde{g})} & & \Big\downarrow{\scriptstyle \underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{g})} & & \\ 0 & \longrightarrow & \underline{\underline{\Gamma}}_{\mathcal{S}_{\mathcal{C}'}}(\tilde{\mathbf{E}}) & \longrightarrow & \underline{\underline{\Gamma}}_{\mathcal{S}_{\mathcal{C}}}(\tilde{\mathbf{E}}) & \longrightarrow & \underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{\mathbf{E}}) & \longrightarrow & 0 \end{array}$$

Again applying induction, it is now sufficient to show that $(\mathrm{Tot}\circ \underline{\underline{\Gamma}}_{Y}\circ \underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}})(\tilde{g})$ is a quasi-isomorphism.

Now let $\underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{\mathbf{F}})'$ be obtained from $\underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{\mathbf{F}})$ by replacing all $d_{(U,Q),(U',Q')}$ where $\dim U' - \dim U \neq 0,1$ by 0. Similarly let $\underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{g})'$ be obtained from $\underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{g})$ by replacing all $g_{(U,Q),(U',Q')}$ where $\dim U \neq \dim U'$ by 0.

Then it follows from the definitions of $\mathbf{F}$ and $\tilde{g}$ that $\underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\mathbf{F})'$ and $\underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{g})'$ are still in $C(\mathcal{T},\mathcal{A})$, and yield identical images in $C(\mathcal{T},K(\mathcal{A}))$ as $\underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{f})$ and $\underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{g})$ under For.

By Theorem 4.5.2 this means there are homotopy invertible maps ϕ, ϕ' in $C(\mathcal{T},\mathbb{Z}_l\text{-mod})$ such that the diagram below is commutative up to homotopy

$$\begin{array}{ccc}
\underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{\mathbf{F}})' & \xrightarrow{\phi} & \underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{\mathbf{F}}) \\
\Big\downarrow{\scriptstyle \underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{g})'} & & \Big\downarrow{\scriptstyle \underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{g})} \\
\underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{\mathbf{F}})' & \xrightarrow{\phi'} & \underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{\mathbf{F}})
\end{array}$$

Hence it is now sufficient to show that $(\mathrm{Tot} \circ \underline{\underline{\Gamma}}_Y)(\underline{\underline{\Gamma}}_{\mathcal{S}_{(P_0,Q_0)}}(\tilde{g})')$ is a quasi-isomorphism.

Since now Tot is merely applying $\pi_{Q,G*}$, and everything is sight is acyclic for $\pi_{Q,G*}$, we are reduced to showing the acyclicity of the simple complex, associated to the following double complex

(5.10)

$$\cdots \xrightarrow{d} \bigoplus_{\substack{\sigma\in\mathcal{P}^{\circ}_{P_0} \\ \dim\sigma+l(Q_0/B)=p-1}} \underline{\underline{\Gamma}}_{G\times^{Q_0}X_\sigma}(G\times^{Q_0}X, I_{Q_0}) \xrightarrow{d} \bigoplus_{\substack{\sigma\in\mathcal{P}^{\circ}_{P_0} \\ \dim\sigma+l(Q_0/B)=p}} \underline{\underline{\Gamma}}_{G\times^{Q_0}X_\sigma}(G\times^{Q_0}X, I_{Q_0}) \xrightarrow{d} \cdots$$
$$\xrightarrow{\epsilon} \underline{\underline{\Gamma}}_{G\times^{Q_0}X_{P_0}}(G\times^{Q_0}X, I_{Q_0}) \to 0$$

Here d is given by $\oplus_{\sigma' \text{ face of } \sigma}\alpha_{\sigma,\sigma'}i_{\sigma',\sigma}$ and ϵ is given by $\oplus\beta_\sigma i_{\sigma,A_{Q_0}}$ where for $U,U' \in A_{Q_0}$, $U \subset U'$ we have used the notation $i_{U,U'}$ for the inclusion

$$\underline{\underline{\Gamma}}_{G\times^{Q_0}X_U}(G\times^{Q_0}X, I_{Q_0}) \to \underline{\underline{\Gamma}}_{G\times^{Q_0}X_{U'}}(G\times^{Q_0}X, I_{Q_0})$$

The proof of the exactness of (5.10) is exactly the same as that of [29, lemmas 3.2.2]

6. Calculation of the spectral sequence (5.7) under condition (*)

In this section we keep the notations of the previous sections and we will assume throughout that condition (*) holds. Under this hypothesis we will compute the E^1 and the E^2-terms of the spectral sequence (5.7) and we will show that it degenerates at E^2.

First we have to introduce some more notations. e_λ will be the codimension of X_λ in X. If λ, $\lambda' \in Y(T)_\mathbb{R}$ the we will say that $\lambda \sim \lambda'$ if $X_\lambda = X_{\lambda'}$. This equivalence relation is clearly $\mathcal{W}_G$-equivariant. If $U \subset Y(T)_\mathbb{R}$ then $U_\lambda = \{\lambda' \in U \mid \lambda' \sim \lambda\}$. U_λ is a locally closed subset of U and it is convex if U is convex.

Lemma 6.1. *Let $\lambda \in C_B$. Then*

$$P = \{g \in G \mid gX_\lambda = X_\lambda\}$$

is a parabolic subgroup of G and it is of the form $P_{\lambda'}$ for some $\lambda' \in (C_B)_\lambda$.

Proof . P is a parabolic since it contains B. Let $\mathcal{W}_P$ be the Weyl group of P. Then $\mathcal{W}_P X_\lambda = X_\lambda$ or $\forall w \in \mathcal{W}_P : w\lambda \sim \lambda$. Hence C_λ is $\mathcal{W}_P$ invariant. Put $\lambda' = \frac{1}{|\mathcal{W}_P|}\sum_{w\in\mathcal{W}_P} w\lambda$. Since C_λ is convex $\lambda' \in C_\lambda$.

We claim that $P = P_{\lambda'}$. To this end, we have to show that for every root ρ of P one has $\langle\lambda', \rho\rangle \geq 0$. But

$$\langle\lambda', \rho\rangle = \frac{1}{|\mathcal{W}_P|}\sum_{w\in\mathcal{W}_P} \langle\lambda, w\rho\rangle$$

Now assume that ρ is a root of the Levy subgroup of P. In that case $\sum_{w\in\mathcal{W}_P} w\rho = 0$ and hence $\langle\lambda', \rho\rangle = 0$.

On the other hand if ρ is a root of the unipotent part of P then all $(w\rho)_{w\in\mathcal{W}_P}$ are roots of B. Since $\lambda \in C_B$ this implies that $\langle\lambda, w\rho\rangle \geq 0$. Hence $\langle\lambda', \rho\rangle \geq 0$.

The fact that $B \subset P = P_\lambda$ implies that $\lambda' \in C_B$. Hence $\lambda' \in C_B \cap C_\lambda = (C_B)_\lambda$.

Clearly the parabolic $P_{\lambda'}$ constructed in the above lemma is the largest parabolic in the set $(P_\mu)_{\mu\in(C_B)_\lambda}$. Note that the existence of such a maximal element was not entirely obvious.

We will choose a set of representants $\Lambda \subset C_B$ for the equivalence classes $C_B/\sim$ in such a way that if $\lambda \in \Lambda$ then $P_\lambda \supset P_\mu$ for all $\mu \sim \lambda$, $\mu \in C_B$. According to lemma 6.1 this is possible.

In the sequel we assume that the roots of B are the negative roots. Φ, Φ^+, S will resp. be the roots, the positive roots and the simple roots of G. If $w \in \mathcal{W}_G$ then $l(w)$ is the length of w with respect to S.

If $\lambda \in C$ then H_λ will be the Levy subgroup of P_λ. and we denote by $\mathcal{W}_\lambda$, Φ_λ, Φ_λ^+, S_λ resp the Weyl group of H_λ (i.e. the stabilizer of λ in $\mathcal{W}_G$), the roots of H_λ (i.e. those roots such that $\langle\lambda, \rho\rangle = 0$), the positive roots of H_λ and the simple roots of H_λ. For $Q \in \mathcal{Q}$ we let $\mathcal{W}_{\lambda,Q}$ be those elements of $\mathcal{W}_\lambda$ which map the positive root of $Q \cap H_\lambda$ inside Φ_λ^+. Note that is $\lambda = 0$ then $P_\lambda = H_\lambda = G$.

We need the following result. Let $Q, Q' \in \mathcal{Q}$ such that $Q \subset Q'$ and let maps be named as in the following diagram :

$$\begin{tikzcd}
G/Q \arrow[r, "\pi_{Q,Q'}"] \arrow[d, "\pi"] & G/Q' \arrow[d, "\pi"] \\
\operatorname{Spec} k \arrow[r, equal] & \operatorname{Spec} k
\end{tikzcd}$$

Note that $\mathcal{W}_{0,Q'} \subset \mathcal{W}_{0,Q}$.

Lemma 6.2. *Let $i \in \mathbb{N}$ and $Q, Q' \in \mathcal{Q}$. Then $R^{2i+1}\pi_*(\mathbb{Z}_l)_{G/Q} = 0$ and $R^{2i}\pi_*(\mathbb{Z}_l)_{G/Q}(i)$ is a free $\mathbb{Z}_l$-module (with trivial Galois action) indexed by those elements of $\mathcal{W}_{0,Q}$ having length $\dim G/Q - i$.*

Furthermore, for this basis, the trace map

$$\pi_*(\mathrm{Tr}_{\pi_{Q,Q'}}) : R^{2i}\pi_*\mathbb{Z}_l(i) \to R^{2(i-\dim Q'/Q}\pi_*\mathbb{Z}_l(i - \dim Q'/Q)$$

is induced by the map $\mathcal{W}_{0,Q} \to \mathcal{W}_{0,Q'}$ which is the identity on $\mathcal{W}_{0,Q'} \subset \mathcal{W}_{0,Q}$, and zero elsewhere.

Proof . This is well know and easy to prove. See [4] as a classical reference for the topological case. The point is that $G/Q = \bigcup_{w\in\mathcal{W}_{0,Q}}(BwQ/Q)$ and $R^{2i}\pi_*\mathbb{Z}_l(i)$ is generated by the characteristic classes of the Bruhat cells $\overline{BwQ/Q}$ of dimension $\dim G/Q - i$.

The functorial properties of the trace map insure that these characteristic classes are compatible with it.

The choice of the set Ξ and the CW-complex $\mathcal{P}$ on C (see §4,2) guarantee that for every $\sigma \in \mathcal{P}_B$ there exist a $\lambda \in \operatorname{relint}\sigma$ such that $X_\lambda = X_\sigma$ and then there is a *unique* $\lambda' \in \Lambda$, $\lambda' \sim \lambda$. This shows that the spectral sequence (5.7) is build up from the following basic building blocks.

$$E^{(q)}_{\lambda,Q} = {}^{\mathrm{perv}}R^{q+2\dim G/Q}(\pi_{Q,G} \circ \underline{\underline{\Gamma}}_{G\times^Q X_\lambda})(\mathbb{Q}_l)_{G\times^Q X}(\dim G/Q)$$

where $\lambda \in \Lambda$, $Q \in \mathcal{Q}$, $Q \subset P_\lambda$. Note that we did switch to $\mathbb{Q}_l$-coefficients.

We will use maps as named in the following diagram :

$$\begin{tikzcd}
e = [P_\lambda] \arrow[d] & P_\lambda/Q \arrow[l, "\pi_{Q,P_\lambda,e}"'] \arrow[d] & \\
G/P_\lambda & G/Q \arrow[l, "\pi_{Q,P_\lambda}"'] & \\
G \times^{P_\lambda} X_\lambda \arrow[u, "f_{P_\lambda}"'] \arrow[d, "\pi_{(\lambda,P_\lambda),G}"'] & G \times^Q X_\lambda \arrow[l, "\pi_{(\lambda,Q),(\lambda,P_\lambda)}"'] \arrow[u, "f_Q"'] \arrow[r, "i"] \arrow[d, "\pi_{(\lambda,Q),G}"'] & G \times^Q X \arrow[d, "\pi_{Q,G}"'] \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}$$

Lemma 6.3. *Assume that condition (*) holds. Then*

(6.1)
$$E^{(q)}_{\lambda,Q} = \begin{cases} 0 & \textit{if } q \not\equiv \dim G \times^{P_\lambda} X_\lambda \bmod 2 \\ \mathcal{G}_\lambda(-\frac{1}{2}q + \frac{1}{2}\dim G \times^{P_\lambda} X_\lambda) \otimes_{\mathbb{Q}_l} B^{(q)}_{\lambda,Q} & \textit{otherwise} \end{cases}$$

where $\mathcal{G}_\lambda$ is a simple perverse sheaf in $D^b_c(X, \mathbb{Q}_l)$ given by

$$\mathcal{G}_\lambda = {}^{perv}R^{\dim G\times^{P_\lambda}X_\lambda}\pi_{(\lambda,P_\lambda),G*}(\mathbb{Q}_l)_{G\times^{P_\lambda}X_\lambda}$$

and $B^{(q)}_{\lambda,Q}$ is the $\mathbb{Q}_l$-vector space with basis

$$\{w \in \mathcal{W}_{\lambda,Q} \mid l(w) = \frac{1}{2}\dim X_\lambda - \frac{1}{2}\dim G/P_\lambda - \frac{1}{2}q + e_\lambda\} \tag{6.2}$$

Proof . First of all note that i is a closed immersion of smooth varieties. Hence we may invoke [1, XVI 3.8,3.10] to rewrite $E^{(q)}_{\lambda,Q}$ as

$$E^{(q)}_{\lambda,Q} = {}^{\mathrm{perv}}R^{q+2\dim G/Q-2e_\lambda}\pi_{(\lambda,Q),G*}(\mathbb{Q}_l)_{G\times^Q X_\lambda}(\dim G/Q - e_\lambda) \tag{6.3}$$

Here

$$\begin{aligned} R\pi_{(\lambda,Q),G*}(\mathbb{Q}_l)_{G\times^Q X_\lambda} &= (R\pi_{(\lambda,P_\lambda),G*}\circ R\pi_{(\lambda,Q),(\lambda,P_\lambda)*})(\mathbb{Q}_l)_{G\times^Q X} \\ &= (R\pi_{(\lambda,P_\lambda),G*}\circ f^*_{P_\lambda})(R\pi_{Q,P_\lambda *}(\mathbb{Q}_l)_{G/Q}) \end{aligned} \tag{6.4}$$

Now by Deligne's criterion [13, Thm 1.5]

$$R\pi_{Q,P_\lambda *}(\mathbb{Q}_l)_{G/Q} = \bigoplus_{i=0}^{\dim P_\lambda/Q} R^i\pi_{Q,P_\lambda *}(\mathbb{Q}_l)_{G/Q}[-i] \tag{6.5}$$

Since $(G/P_\lambda)_{\overline{k}}$ is simply connected, we may compute the righthand side of (6.5) in a rational point. We choose $e=[P_\lambda]$ for this rational point. We find that

$$(6.5) = \bigoplus_{i=0}^{2\dim P_\lambda/Q} (R^i\pi_{Q,P_\lambda,e*}((\mathbb{Q}_l)_{P_\lambda/Q})\otimes_{\mathbb{Q}_l}(\mathbb{Q}_l)_{G/P_\lambda})[-i] \tag{6.6}$$

To simplify the notation a bit, we will put

$$A^{(i)}_{\lambda,Q} = R^{2i}\pi_{Q,P_\lambda,e*}(\mathbb{Q}_l)(i)$$

$A^{(i)}_{\lambda,Q}$ may be identified with a $\mathbb{Q}_l$ vector space with trivial $\mathrm{Gal}(\overline{k}/k)$ action. Hence we find

$$R\pi_{Q,P_\lambda *}(\mathbb{Q}_l)_{G/Q} = \bigoplus_{i=0}^{\dim P_\lambda/Q}(A^{(i)}_{\lambda,Q}\otimes_{\mathbb{Q}_l}(\mathbb{Q}_l)_{G/P_\lambda})(-i)[-2i]$$

Substituting this in (6.4) yields

$$R\pi_{(\lambda,Q),G*}(\mathbb{Q}_l)_{G\times^Q X_\lambda} = \bigoplus_{i=0}^{\dim P_\lambda/Q}(R\pi_{(\lambda,P_\lambda),G*}(\mathbb{Q}_l)_{G\times^{P_\lambda}X_\lambda}\otimes_{\mathbb{Q}_l}A^{(i)}_{\lambda,Q})(-i)[-2i]$$

Now by Proposition 3.2.1 and condition (*)

$$\mathcal{G}_\lambda = R\pi_{(\lambda,P_\lambda),G*}\mathbb{Q}_l[\dim G\times^{P_\lambda}X_\lambda]$$

is a simple perverse sheaf. Hence we find that

$$R\pi_{(\lambda,Q),G*}(\mathbb{Q}_l)_{G\times^Q X_\lambda} = \mathcal{G}_\lambda\otimes\left(\bigoplus_{i=0}^{\dim P_\lambda/Q}A^{(i)}_{\lambda,Q}(-i)[-2i-\dim G\times^{P_\lambda}X_\lambda]\right) \tag{6.7}$$

A summand in (6.7) will not contribute to (6.3) unless

$$i=\phi^{(q)}_{\lambda,Q}$$

where $\phi^{(q)}_{\lambda,Q}$ is the following magic number

$$\phi^{(q)}_{\lambda,Q} = \frac{1}{2}q+\dim G/Q - e_\lambda - \frac{1}{2}\dim G\times^{P_\lambda}X_\lambda$$

To make the notation less heavy, we put $B^{(q)}_{\lambda,Q} = A^{(*)}_{\lambda,Q}$ where $* = \phi^{(q)}_{\lambda,Q}$ if $\phi^{(q)}_{\lambda,Q}$ is integral. Then combining (6.7) and (6.3) yields (6.1).

According to lemma 6.2

$$B^{(q)}_{\lambda,Q} = \{w\in\mathcal{W}_{\lambda,Q}\mid l(w) = \dim P_\lambda/Q - \phi^{(q)}_{\lambda,Q}\}$$

which yields (6.2).

Lemma 6.4. *Assume condition (*), $q \cong q' \cong \dim G \times^{P_\lambda} X_\lambda \bmod 2$ and k finitely generated over $\mathbb{Q}$. Then for $Q, Q' \in \mathcal{Q}$, $Q, Q' \subset P_\lambda$*

$$\mathrm{Hom}_{D^b_c(X,\mathbb{Q}_l)}(E^{(q)}_{\lambda,Q}, E^{(q')}_{\lambda',Q'}) = 0 \tag{6.8}$$

unless $\lambda = \lambda'$, $q = q'$. In that case

$$\mathrm{Hom}_{D^b_c(X,\mathbb{Q}_l)}(E^{(q)}_{\lambda,Q}, E^{(q)}_{\lambda,Q'}) = \mathrm{Hom}_{\mathbb{Q}_l}(B^{(q)}_{\lambda,Q}, B^{(q)}_{\lambda,Q'}) \tag{6.9}$$

Furthermore the trace morphism for $Q \subset Q'$

$$R\pi_{Q',G*}(\mathrm{Tr}_{\pi_{Q,Q'}}) : E^{(q)}_{\lambda,Q} \to E^{(q)}_{\lambda,Q'}$$

corresponds, under the identification (6.2) *to the natural projection*

$$\mathcal{W}_{\lambda,Q} \to \mathcal{W}_{\lambda,Q'}$$

which is the identity on $\mathcal{W}_{\lambda,Q'} \subset \mathcal{W}_{\lambda,Q}$ and zero otherwise.

Proof . We use the fact that the $\mathcal{G}_\lambda$ are simple perverse sheaves, with support GX_λ. I.e. $\mathrm{Hom}(\mathcal{G}_\lambda, \mathcal{G}_{\lambda'}) = 0$ if $\lambda \neq \lambda'$ using condition (*), and consequently (6.8) is true if $\lambda \neq \lambda'$. Hence assume $\lambda = \lambda'$. Then

$$\mathrm{Hom}_{D^b_c(X,\mathbb{Q}_l)}(E^{(q)}_{\lambda,Q}, E^{(q)}_{\lambda,Q'}) = \mathrm{Hom}_{\mathbb{Q}_l}(B^{(q)}_{\lambda,Q}, B^{(q')}_{\lambda,Q'}) \otimes_{\mathbb{Q}_l} \Gamma(\mathbb{Q}_l(-\frac{1}{2}(q-q'))$$

Since we are over a finitely generated extension of $\mathbb{Q}$, $\Gamma(\mathbb{Q}_l(-\frac{1}{2}(q-q'))$ is zero unless $q = q'$ in which case it is $\mathbb{Q}_l$. This proves the first half of lemma 6.4.

To prove the second half assume $Q \subset Q'$. We remember that $B^{(q)}_{\lambda,Q}$ was an abbreviation for

$$R^{2\phi^{(q)}_{\lambda,Q}}\pi_{Q,P_\lambda,e}(\mathbb{Q}_l)_{P_\lambda/Q}(\phi^{(q)}_{\lambda,Q})$$

Since $\phi^{(q)}_{\lambda,Q'} = \phi^{(q)}_{\lambda,Q} - \dim Q'/Q$ there is a trace map

$$R\pi_{Q',P_\lambda,e*}(\mathrm{Tr}_{\pi_{Q,Q',e}}) : R^{2\phi^{(q)}_{\lambda,Q}}\pi_{Q,P_\lambda,e*}(\mathbb{Q}_l)_{P_\lambda/Q}(\phi^{(q)}_{\lambda,Q}) \to R^{2\phi^{(q)}_{\lambda,Q'}}\pi_{Q',P_\lambda,e*}(\mathbb{Q}_l)_{P_\lambda/Q'}(\phi^{(q)}_{\lambda,Q'})$$

and by lemma 6.2 this map is precisely induced from the projection $\mathcal{W}_{\lambda,Q} \to \mathcal{W}_{\lambda,Q'}$.

We claim that this map corresponds to $E^{(q)}_{\lambda,Q} \to E^{(q)}_{\lambda,Q'}$. This follows by following the computations in the proof of lemma 6.3 using the usual properties of the trace map such as lemma 3.2.2, compatibility with base change, and with compositions of maps. The argument, which uses the maps indicated in the following diagram, but notationally somewhat awkward.

$$\begin{tikzcd}
P_\lambda/Q \arrow[r] \arrow[d, "\pi_{Q,Q',e}"] & G/Q \arrow[d] & G \times^Q X_\lambda \arrow[l] \arrow[r] \arrow[d] & G \times^Q X \arrow[d] \\
P_\lambda/Q' \arrow[r] \arrow[d] & G/Q' \arrow[d] & G \times^{Q'} X_\lambda \arrow[l] \arrow[r] \arrow[d] & G \times^{Q'} X \arrow[d] \\
e = [P_\lambda] \arrow[r] & G/P_\lambda & G \times^{P_\lambda} X_\lambda \arrow[l] \arrow[r] \arrow[d] & G \times_{P_\lambda} X \arrow[d] \\
 & & X \arrow[r, equal] & X
\end{tikzcd} \tag{6.10}$$

□

Lemma 6.5. *Assume condition (*).*

(1) *The spectral sequence (5.7), with $\mathbb{Q}_l$-coefficients, degenerates at the E^2-term.*
(2) *The E^1-term, with $\mathbb{Q}_l$-coefficients, has the form*

$$E^1_{-pq} = \bigoplus_{\lambda\in\Lambda} \mathcal{G}_\lambda(-\frac{1}{2}q + \frac{1}{2}\dim G\times^{P_\lambda} X_\lambda)\otimes_{\mathbb{Q}_l} E^1_{-pq,\lambda}$$

where

$$E^1_{-pq,\lambda} = \begin{cases} 0 & \text{if } q \not\equiv \dim G\times^{P_\lambda} X_\lambda \bmod 2 \\ \bigoplus_{\substack{\sigma\in\mathcal{P}_B, Q\in\mathcal{Q} \\ \operatorname{relint}\sigma\subset(C_Q\backslash\partial C)_\lambda \\ \dim\sigma-\dim T+l(Q/B)=-p}} B^{(q)}_{\lambda,Q} & \text{otherwise} \end{cases}$$

Furthermore the differential $d_{-pq}: E^1_{-pq}\to E^1_{-p+1,q}$ is induced from differentials on $E^1_{-pq,\lambda}$ of the form

$$\bigoplus \alpha_{(\sigma',Q'),(\sigma,Q)} p^{(q)}_{\lambda,Q,Q'}$$

where $p^{(q)}_{\lambda,Q,Q'}$ stands now for the map $B^{(q)}_{\lambda,Q}\to B^{(q)}_{\lambda,Q'}$, obtained from the natural projection $\mathcal{W}_{\lambda,Q}\to\mathcal{W}_{\lambda,Q'}$

Proof . We may assume that k is finitely generated over $\mathbb{Q}$. The the lemma is a direct combination of lemmas 6.3 and 6.4 and since $\operatorname{Hom}(\mathcal{G}_\lambda(-\frac{1}{2}q+\frac{1}{2}\dim G\times^{P_\lambda} X_\lambda), \mathcal{G}_{\lambda'}(-\frac{1}{2}q'+\frac{1}{2}\dim G\times^{P_{\lambda'}} X_{\lambda'})) = 0$ unless $\lambda=\lambda'$, $q=q'$.

Now $E^1_{-pq,\lambda}$ may be simplified further.

Lemma 6.6.

$$E^1_{-pq,\lambda} = \bigoplus_{w\in\mathcal{W}_\lambda} E^1_{-pq,w,\lambda}$$

with

$$E^1_{-pq,w,\lambda} = \begin{cases} \mathbb{Q}_l\mathcal{U}_{-p,w,\lambda} & \text{if } q = \dim X_\lambda - \dim G/P_\lambda - 2l(w) + 2e_\lambda \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{U}_{-p,w,\lambda}$ is the set

$$\{(\sigma,Q)\mid \sigma\in\mathcal{P}_B, Q\in\mathcal{Q}, w\in\mathcal{W}_{\lambda,Q}, \operatorname{relint}\sigma\subset(C_Q\backslash\partial C)_\lambda, \dim\sigma-\dim T+l(Q/B)=-p\}$$

The differential $d_{-pq}: E_{-pq}\to E_{-p+1,q}$ induces differentials on $\mathbb{Q}_l\mathcal{U}_{-p,w,\lambda}$ given by

$$d(\sigma,Q) = \sum_{(\sigma',Q')} \alpha_{(\sigma',Q'),(\sigma,Q)}(\sigma',Q')$$

Here (σ',Q') runs through $\mathcal{U}_{-p+1,q,w,\lambda}$ with $\sigma\subset\sigma'$, $Q\subset Q'$, $\dim\sigma'=\dim\sigma+1$, $l(Q'/Q)=0$, or $\dim\sigma'=\dim\sigma$ and $l(Q'/Q)=1$.

Proof . This is immediate from lemma 6.5

Hence we have to compute the homology of

$$(\mathbb{Q}_l\mathcal{U}_{*,w,\lambda})_*$$

First we introduce a few lemmas which will be used afterwards.

Lemma 6.7. *Let $w\in\mathcal{W}_\lambda$, $\lambda\in C_B$. Then there exists a unique $Q\in\mathcal{Q}$, $Q\subset P_\lambda$ (denoted by $P_{w,\lambda}$ below) which is maximal for the property $w\in\mathcal{W}_{\lambda,Q}$.*

Proof . Let $Q \in \mathcal{Q}$, $Q \subset P_\lambda$, $w \in \mathcal{W}_\lambda$. Denote the simple and the positive roots of Q by respectively S_Q and Φ_Q^+. If $S \subset X(T)$ write $\langle S\rangle^+$ for the positive integer linear combinations of S. Then $\Phi_Q^+ = \langle S\rangle^+ \cap \Phi_\lambda$.

We have $w \in \mathcal{W}_{\lambda,Q} \iff w\Phi_Q^+ \subset \Phi_\lambda^+ \overset{(1)}{\iff} wS_Q \subset \Phi_\lambda^+ \iff S_Q \subset S_\lambda \cap w^{-1}\Phi_\lambda^+$.
$\overset{(1)}{\Leftarrow}$ is seen as follows : assume $wS_Q \subset \Phi_\lambda^+$. Then $w\Phi_Q^+ = \langle wS_Q\rangle^+ \cap \Phi_\lambda \subset \langle \Phi_\lambda\rangle^+ \cap \Phi_\lambda = \Phi_\lambda^+$.
It now follows that the maximal case is given by $S_Q = S_\lambda \cap w^{-1}\Phi_\lambda^+$

For $\sigma \in \mathcal{P}_B$ denote by P_σ the largest element of $\mathcal{Q}$ such that $\operatorname{relint}\sigma \subset A_{P_\sigma}$.

Lemma 6.8. *Let $\lambda \in C_B$, $w \in \mathcal{W}_\lambda$. Let $A_B^{(w,\lambda)} = A_B - \bigcup_{s\in S_\lambda \cap w^{-1}\Phi_\lambda^+} A_{P_s}$ where we let P_s stand for the parabolic containing B and having s as a unique simple root. Then $A_B^{(w,\lambda)}$ has the property that*

$$\forall \sigma \in \mathcal{P}_B : P_\sigma \cap P_{w,\lambda} = B \iff \operatorname{relint}\sigma \subset A_B^{(w,\lambda)}$$

Proof . For $U \subset A_B$, $V \subset X(T)_\mathbb{R}$ we denote

$$V^{\perp U} = \{v \in V \mid \forall u \in U, \langle u, v\rangle = 0\}$$

and a similar definition for $U^{\perp V}$.

For $P \in \mathcal{Q}$ let us denote by S_P the simple roots. By the proof of lemma 6.7

$$S_{P_{w,\lambda}} = S_\lambda \cap w^{-1}\Phi_\lambda^+$$

Hence the condition

$$P_\sigma \cap P_{w,\lambda} = B$$

may be reformulated as

$$S_{P_\sigma} \cap (S_\lambda \cap w^{-1}\Phi_\lambda^+) = \emptyset \tag{6.11}$$

Now $S_{P_\sigma} = S^{\perp\sigma}$ and hence (6.11) may be rewritten as

$$(S_\lambda \cap w^{-1}\Phi_\lambda^+)^{\perp\sigma} = \emptyset \tag{6.12}$$

Let $\lambda' \in \operatorname{relint}\sigma$. By our construction of $\mathcal{P}_B$ (see §5.2) (6.12) is equivalent with

$$(S_\lambda \cap w^{-1}\Phi_\lambda^+)^{\perp\lambda'} = \emptyset$$

Or

$$\lambda' \in A_B - \bigcup_{s\in S_\lambda \cap w^{-1}\Phi_\lambda^+} A_B^{\perp s}$$

which shows what we want.

Now we are ready to state the main result of this section. Let us call a pair $(w, \lambda) \in \mathcal{W}_G \times \Lambda$ admissible if $w \in \mathcal{W}_\lambda$ and if

$$(A_B)_\lambda \cap A_B^{(w,\lambda)} \neq \emptyset$$

For (w, λ) admissible, define

$$\Psi_{w,\lambda} = \overline{(C_B \setminus \partial C)_\lambda \cap A_B^{(w,\lambda)}} - (C_B \setminus \partial C)_\lambda \cap A_B^{(w,\lambda)}$$

It is easy to see that $(C_B \setminus \partial C)_\lambda \cap A_B^{(w,\lambda)}$ is can be written as the intersection of an open and a closed set. This implies that $\Psi_{w,\lambda}$ is closed in $Y(T)_\mathbb{R}$.

If $\lambda \in \Lambda$ put $f_\lambda = \operatorname{codim} GX_\lambda$. Note that under condition (*) $f_\lambda = e_\lambda - \dim G/P_\lambda$.

Theorem 6.9. *Assume that condition (*) holds. Then* ${}^{perv}\mathcal{H}^n_{X^u}(X, \mathbb{Q}_l[\dim X])$ *is filtered, with associated graded quotients*

$$\bigoplus_{(w,\lambda)\ admissible} \tilde{H}^{n+\dim T - f_\lambda + 2l(w)-1}(\Psi_{w,\lambda}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathcal{G}_\lambda(l(w) - f_\lambda)$$

Here $\mathcal{G}_\lambda$ *is the simple perverse sheaf*

$$R\pi_{(\lambda,P_\lambda),G*}(\mathbb{Q}_{lG\times^{P_\lambda} X_\lambda}[\dim G \times^{P_\lambda} X_\lambda])$$

Proof . According to lemma 6.6, we have to compute the homology of $(\mathbb{Q}_l\mathcal{U}_{*,w,\lambda})_*$.

First of all, note that one may rewrite $\mathcal{U}_{-p,w,\lambda}$ as

$$\{(\sigma, Q) \mid \sigma \in \mathcal{P}_B, \operatorname{relint}\sigma \subset (C_B \backslash \partial C)_\lambda, Q \in \mathcal{Q}, Q \subset P_\sigma \cap P_{w,\lambda}, \dim\sigma - \dim T + l(Q/B) = -p\}$$

Now we may filter $(\mathbb{Q}_l\mathcal{U}_{*,w,\lambda})_*$ according to $\dim\sigma$ and then the associated graded complexes are direct sums of reduced cochain complexes of abstract complexes of the form

$$\{Q \in \mathcal{Q} \mid B \subset Q \subset P_\sigma \cap P_{w,\lambda}\}$$

Hence these are acyclic, unless $P_\sigma \cap P_{w,\lambda} = B$.

Hence $(\mathbb{Q}_l\mathcal{U}_{*,w,\lambda})_*$ is quasi-isomorphic to its quotient complex $(\mathbb{Q}_l\mathcal{U}'_{*,w,\lambda})_*$ where

$$\begin{aligned}\mathcal{U}'_{-p,w,\lambda} &= \{\sigma \in \mathcal{P}_B \mid \operatorname{relint}\sigma \subset (C_B \setminus \partial C)_\lambda, P_\sigma \cap P_{w,\lambda} = B, \dim\sigma - \dim T = -p\} \\ &= \{\sigma \in \mathcal{P}_B \mid \operatorname{relint}\sigma \subset (C_B \setminus \partial C)_\lambda \cap A_B^{(w,\lambda)}, \dim\sigma = \dim T - p\}\end{aligned}$$

Hence if $(C_B \setminus \partial C)_\lambda \cap A_B^{(w,\lambda)} = \emptyset$ then $\mathcal{U}'_{*,w,\lambda} = \emptyset$ and there is no homology

If $(C_B \setminus \partial C)_\lambda \cap A_B^{(w,\lambda)} \neq \emptyset$ then the homology of $(\mathbb{Q}_l\mathcal{U}'_{*,w,\lambda})_*$ is equal to

$$H^{\dim T - p}(\overline{(C_B \setminus \partial C)_\lambda \cap A_B^{(w,\lambda)}}, \Psi_{w,\lambda}, \mathbb{Q}_l) = \tilde{H}^{\dim T - p - 1}(\Psi_{w,\lambda}, \mathbb{Q}_l) \tag{6.13}$$

in degree $-p$. (Here we have used that $\overline{(C_B \setminus \partial C)_\lambda \cap A_B^{(w,\lambda)}}$ is convex and hence contractible.) According to lemma 6.5 and lemma 6.6, (6.13) gives a contribution to $\operatorname{gr}{}^{\mathrm{perv}}\mathcal{H}^n_{X^u}(X, \mathbb{Q}_l)$ of the form

$$\mathcal{G}_\lambda(-\frac{1}{2}q + \frac{1}{2}\dim G \times^{P_\lambda} X_\lambda) \otimes \tilde{H}^{\dim T - p - 1}(\Psi_{w,\lambda}, \mathbb{Q}_l) \tag{6.14}$$

where

$$q = \dim X_\lambda - \dim G/P_\lambda - 2l(w) + 2e_\lambda$$

and $n = -p + q$.

Then (6.14) may be rewritten as

$$\mathcal{G}_\lambda(l(w) - f_\lambda) \otimes \tilde{H}^{\dim T - \dim X - f_\lambda + 2l(w) + n - 1}(\Psi_{w,\lambda}, \mathbb{Q}_l)$$

This yields the desired result.

7. Proofs and examples

In this section the ground field will be $\mathbb{C}$. By the Lefschetz principle, the results remain of course valid for any algebraically closed field of char. 0. We keep otherwise the notations of the preceding sections.

7.1. **The description of $H^i_{X^u}(X, \mathcal{O}_X)$ as $(G, \mathcal{D}_X)$-module.** We will apply the results of the previous section to the computation of $H^i_{X^u}(X, \mathcal{O}_X)$ when condition (*) holds. Since X is affine we will silently identify $H^i_{X^u}(X, \mathcal{O}_X)$ and $\mathcal{H}^i_{X^u}(X, \mathcal{O}_X)$.

The main tool will of course be the Riemann-Hilbert correspondence, and we will follow the notations of the standard reference [5]. In particular the De Rham-functor DR(?) will be the ordinary De Rham-functor, suitably shifted in such a way that it sends holonomic modules with regular singularities to perverse sheaves.

Proof. Proof of Theorem 2.1 First note that it was shown in §3.1 that $H^n_{X^u}(X, \mathcal{O}_X)$ and $\mathcal{L}(GX_\lambda, X)$ are in $(G, \mathcal{D})$-qch. Then by Proposition 3.1.2 it suffices to prove (2.1) without the G-structure. Let $\mathcal{D}_X$-rhol be the category of holonomic $\mathcal{D}_X$-modules with regular singularities. Since

$$\mathrm{DR} : D^b(\mathcal{D}_X\text{-rhol}) \to D^b_c(X(\mathbb{C}), \mathbb{C})$$

commutes with the usual cohomology operations [5, VIII, 14.5], it follows that

$$\mathrm{DR}(H^n_{X^u}(X, \mathcal{O}_X)) = {}^{\mathrm{perv}}\mathcal{H}^n_{X^u}(X(\mathbb{C}), \mathbb{C})[\dim X]$$

There are functors

$$D^b_c(X, \mathbb{Q}_l) \to D^b_c(X(\mathbb{C}), \mathbb{Q}_l) \to D^b_c(X(\mathbb{C}), \mathbb{C})$$

The first one is obtained from the morphism of toposes $X(\mathbb{C}) \to X_{\mathrm{et}}$ [2, §6.1.2][14, §5] and the second one is extension of the coefficient field. One verifies that these functors commute with the usual cohomology operations and hence that they commute with perverse homology. Hence to compute ${}^{\mathrm{perv}}\mathcal{H}^n_{X^u}(X(\mathbb{C}), \mathbb{C})$ it suffices to compute ${}^{\mathrm{perv}}\mathcal{H}^n_{X^u}(X, \mathbb{Q}_l)$, which is done in Theorem 6.9.

Since $G \times^{P_\lambda} X_\lambda \to GX_\lambda$ is small, the $\mathcal{G}_\lambda$ are intersection homology perverse sheaves (Proposition 3.2.1). Hence via the Riemann-Hilbert correspondence, they must correspond to $\mathcal{L}(GX_\lambda, X)$. □

7.2. **When does condition (*) hold.** In contrast to the torus case, (*) is not always true, and furthermore it is easy to see that Theorem 2.1 is false if (*) does not hold.

In this section we give some "stable" criteria for (*) to hold. The first one says that (*) is true if the irreducible subrepresentations of W occur with high enough multiplicities. The second one, for simple groups, asserts that (*) holds if W has a simple subrepresentation, with a big highest weight which lies in addition in the root lattice. As a corollary we obtain that if G is simple of adjoint type then (*) is satisfied for all but a finite number of W.

We start with some preparatory lemmas.

Lemma 7.2.1. *Suppose that $\lambda \in \Lambda$, $w \in \mathcal{W}_G$, $w\lambda \sim \mu$ with μ in A_B. Then $w\lambda = \lambda$.*

Proof . Since $wX_\lambda = X_{w\lambda} = X_\mu$ is B-stable, $BwX_\lambda = wX_\lambda$ or $w^{-1}BwX_\lambda = X_\lambda$ which, by the definition of Λ (after lemma 6.1), implies $w^{-1}Bw \subset P_\lambda$. Consequently $B \subset P_{w\lambda}$ which is only possible if w stabilizes λ.

Lemma 7.2.2. *Let $\lambda \in \Lambda$ and let $\pi : G \times^{P_\lambda} X_\lambda \to GX_\lambda$ be the natural projection map. Then π is one-one at $x \in GX_\lambda \iff$*

$$x \notin \bigcup_{w \in \mathcal{W}_G \setminus \mathcal{W}_\lambda} G(X_\lambda \cap wX_\lambda)$$

Proof . Assume that $x \in X_\lambda \cap wX_\lambda$ for some $w \in \mathcal{W}_G \setminus \mathcal{W}_\lambda$. Choose a representant of w in G, and denote it by w too. Then $\pi(w, w^{-1}x) = x$. Since $w \notin \mathcal{W}_\lambda$, $(w, w^{-1}x) \neq (1, x)$ and hence π is not one-one at x

Conversely let $x \in X_\lambda$ and suppose that π is not one-one at x. I.e. there exist $(g, y) \in G \times X_\lambda$, $g \notin P_\lambda$ such that $x = gy$, i.e. $g^{-1}x \in X_\lambda$.

Now there exist $b_1, b_2 \in B$, $w \in \mathcal{W}_G \setminus \mathcal{W}_\lambda$ such that $g = b_1 w b_2$, i.e. $b_2^{-1} w^{-1} b_1^{-1} x \in X_\lambda$ or $x \in B(X_\lambda \cap wX_\lambda) \subset G(X_\lambda \cap wX_\lambda)$. This shows what we want.

Lemma 7.2.3. *Let λ, $\mu \in \Lambda$. Then $GX_\lambda \subset GX_\mu \iff$ there is some $w \in \mathcal{W}_G$ such that*

$$\dim B(X_\lambda \cap wX_\mu) = \dim X_\lambda$$

Proof . We use the following chain of equivalences

$$GX_\lambda \subset GX_\mu \iff X_\lambda \subset GX_\mu \iff X_\lambda \subset \bigcup_{w \in \mathcal{W}_G} BwX_\mu \overset{(1)}{\iff}$$

$$\exists w \in \mathcal{W}_G : X_\lambda \cap BwX_\mu \text{ dense in } X_\lambda \iff \exists w \in \mathcal{W}_G : \dim B(X_\lambda \cap wX_\mu) = \dim X_\lambda$$

To prove $\overset{(1)}{\Leftarrow}$ one uses that $\bigcup_w BwX_\mu = GX_\mu$ is closed. $\overset{(1)}{\Rightarrow}$ follows from the fact that the BwX_μ are constructible.

Theorem 7.2.4. *There is a number N, depending only on G with the property that, if all irreducible subrepresentations of W have multiplicity $\geq N$ then (*) is true.*

Proof . Assume that $W = V_1^{\oplus n_1} \oplus \cdots \oplus V_m^{\oplus n_m}$, the $(V_i)_i$ irreducible and distinct, and let $n = \min(n_i)_i$.

Put $W' = V_1 \oplus \cdots \oplus V_m$. Below we will denote with a prime constructions that that relate to W' instead of to W. In particular one may define Λ', but is easy to see that $\Lambda' = \Lambda$.

For $G \times^{P_\lambda} X_\lambda \to GX_\lambda$ to be a small resolution, it is clearly sufficient by lemma 7.2.2 that $\forall w \in \mathcal{W}_G \setminus \mathcal{W}_\lambda$

$$\dim GX_\lambda - \dim G(X_\lambda \cap wX_\lambda) > 2\dim G/P_\lambda \tag{7.1}$$

since $\dim G/P_\lambda$ is the maximal dimension of a fiber of $G \times^{P_\lambda} X_\lambda \to GX_\lambda$.

(7.1) is clearly implied by

$$\dim X_\lambda - \dim X_\lambda \cap wX_\lambda > 3\dim G \tag{7.2}$$

We will choose N in such a way that (7.2) is fulfilled if $n \geq N$.

First note that by lemma 7.2.1

$$\dim X'_\lambda - \dim X'_\lambda \cap wX'_\lambda \geq 1$$

Hence

$$\begin{aligned}\dim X - X_\lambda \cap wX_\lambda &\geq n(\dim X'_\lambda - \dim(X'_\lambda \cap wX'_\lambda)) \\ &\geq n\end{aligned}$$

Therefore it suffices to take $N = 3\dim G + 1$.

Now let $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$ and assume $GX_\lambda = GX_\mu$. According to lemma 7.2.3 there exist $w, w' \in \mathcal{W}_G$ such that $\dim B(X_\lambda \cap wX_\mu) = \dim X_\lambda$ and $\dim B(X_\mu \cap w'X_\lambda) = \dim X_\mu$.

In particular

$$\begin{aligned} \dim X_\lambda - \dim(X_\lambda \cap wX_\mu) &\leq \dim B \\ \dim X_\mu - \dim(X_\mu \cap w'X_\lambda) &\leq \dim B \end{aligned} \tag{7.3}$$

Now we claim that either

$$\begin{aligned} &\dim X'_\lambda \neq \dim(X'_\lambda \cap wX'_\mu) \qquad \text{or} \\ &\dim X'_\mu \neq \dim(X'_\mu \cap w'X'_\lambda) \end{aligned} \tag{7.4}$$

Suppose that on the contrary both inequalities in (7.4) are equalities. Then $X'_\lambda = wX'_\mu$ but by lemma 7.2.1 this implies $\lambda \sim \mu$, which contradicts the hypotheses.

As above we now conclude

$$\begin{aligned} &\dim X_\lambda - \dim(X_\lambda \cap wX_\mu) \geq n \qquad \text{or} \\ &\dim X_\mu - \dim(X_\mu \cap wX_\lambda) \geq n \end{aligned} \tag{7.5}$$

which, if $n \geq N$, contradicts (7.3).

Now we start with the proof of the second stable criterion

Lemma 7.2.5. *Let $\lambda, \mu \in Y(T)_\mathbb{R}$ and assume that $A_{P_\lambda} \not\subset A_{P_\nu}$. Then there exist $\rho \in \Phi$ such that $\langle \lambda, \rho \rangle < 0$ and $\langle \mu, \rho \rangle \geq 0$.*

Proof . Assume that $\beta \in A_{P_\lambda} \setminus A_{P_\mu}$. Since

$$A_{P_\lambda} = \{\zeta \in Y(T)_\mathbb{R} \mid \forall \rho \in \Phi : \langle \lambda, \rho \rangle \geq 0 \Rightarrow \langle \zeta, \rho \rangle \geq 0\}$$

and a similar statement for A_{P_μ}, we find that

$$\forall \rho \in \Phi : \langle \lambda, \rho \rangle \geq 0 \Rightarrow \langle \beta, \rho \rangle \geq 0$$

and there exist a $\rho' \in \Phi$ such that $\langle \mu, \rho' \rangle \geq 0$ but $\langle \beta, \rho' \rangle < 0$ which implies $\langle \lambda, \rho' \rangle < 0$. Hence ρ' is the sought element of Φ.

Lemma 7.2.6. *Assume that $\mathcal{W}$ is a finite group and E is a finite dimensional irreducible representation of $\mathcal{W}$ over $\mathbb{R}$. Let $(\ ,\ )$ be a $\mathcal{W}$-invariant positive definite bilinear form on E. Then there exist and $r > 0$ such that for any $\lambda \in E$, $(\lambda, \lambda) = 1$, the convex hull of $(w\lambda)_{w \in \mathcal{W}_G}$ contains a closed ball of radius r (with respect to the distance given by $(\ ,\ )$).*

Proof . Let $S \subset E$ be the unit sphere and let B_r stand for a closed ball of radius r. Denote the convex hull of $(w\lambda)_w$ by Γ_λ. First note that for any $\lambda \in E$, 0 lies in the relative interior of Γ_λ since $\lambda' = \frac{1}{|G|} \sum g\lambda$ is G-invariant and since E is irreducible this implies $\lambda' = 0$.

We now define a function

$$\phi : S \to [0, 1] : \lambda \mapsto \max_{B_r \subset \Gamma_\lambda} r$$

It is not hard to verify that ϕ is continuous and since S is compact, ϕ has a minimum which we call r. This is the r we want provided that it is not 0. Suppose that $r = 0$, i.e. there is a λ such that 0 lies on the boundary of Γ_λ. Since 0 also lies in the relative interior, this implies $\dim \Gamma_\lambda < \dim E$. But this means that λ generates a subrepresentation of E, which contradicts our hypotheses.

Below (,) will be a $\mathcal{W}_G$ invariant form on $X(T)_\mathbb{R}$ and $\| \ \|$ will be its corresponding norm.

Theorem 7.2.7. *Let G be simple. Then there exists a real number M, depending only on G, with the property that, if W contains an irreducible representation with highest weight χ lying in the root lattice having the property that $\|\chi\| \geq M$, then (G,W) satifies condition (*).*

Proof . Let W,χ be as in the statement of the theorem. We follow more or less the strategy of the proof of Theorem 7.2.4. Let $\lambda \in \Lambda$.

For $G\times^{P_\lambda} X_\lambda \to GX_\lambda$ to be a small resolution it suffices that $\forall w \in \mathcal{W}_G \setminus \mathcal{W}_\lambda$

$$\dim X_\lambda - \dim(X_\lambda \cap wX_\lambda) > 3\dim G \tag{7.6}$$

Take $w \in \mathcal{W}_G \setminus \mathcal{W}_\lambda$. Since $w\lambda \neq \lambda$ there exist by lemma 7.2.5 a $\rho' \in \Phi$ such that $\langle w\lambda, \rho'\rangle \geq 0$, $\langle \lambda,\rho'\rangle < 0$. Put $\rho = -\rho'$. Hence those $-n\rho$, $n \geq 1$ that are weights of X will also be weights of X_λ, but not of $wX_\lambda = X_{w\lambda}$.

Now the weights of X contain the integral linear combinations of roots lying in the convex hull of $w\chi$. We apply now lemma 7.2.6 with $E = X(T)_\mathbb{R}$ and $\mathcal{W} = \mathcal{W}_G$ and we let r stand for the corresponding number defined in that lemma. We then find that $-n\rho$ will be a weight of X for

$$\frac{-\|\chi\|r}{m} \leq n \leq \frac{\|\chi\|r}{m}$$

where m is $\max_{\rho\in\Phi}\|\rho\|$.

I.e.

$$\dim X_\lambda - \dim(X_\lambda \cap wX_\lambda) > \frac{\|\chi\|r}{m} - 2$$

which implies (7.6) if $\|\chi\| \geq \frac{(3\dim G+2)m}{r}$. I.e. if we put $M = \frac{(3\dim G+2)m}{r}$ and we assume that $\|\chi\| \geq M$ then $G\times^{P_\lambda} X \to GX_\lambda$ will be a small resolution.

Let $\lambda,\mu \in \Lambda$, $\lambda \neq \mu$ and suppose $GX_\lambda = GX_\mu$. I.e. there exist $w \in \mathcal{W}_G$, $w' \in \mathcal{W}_G$ with the property that

$$\begin{aligned} \dim X_\lambda &= \dim B(X_\lambda \cap wX_\mu) \\ \dim X_\mu &= \dim B(X_\mu \cap w'X_\lambda) \end{aligned} \tag{7.7}$$

Clearly, not both $w \in \mathcal{W}_{P_\mu}$ and $w' \in \mathcal{W}_\lambda$. Hence assume $w \notin \mathcal{W}_{P_\mu}$.

(7.7) implies that

$$\begin{aligned} \dim X_\lambda - \dim(X_\lambda \cap wX_\mu) &\leq \dim B \\ \dim X_\mu - \dim(X_\mu \cap w'X_\lambda) &\leq \dim B \end{aligned} \tag{7.8}$$

If we would have that $A_{P_\lambda} \subset A_{P_{w\mu}}$ and $A_{P_\mu} \subset A_{P_{w'\mu}}$ then $A_{P_\lambda} = A_{P_{w\mu}}$ which is impossible since $w\mu \neq \mu$.

Hence we may for example assume that $A_{P_\lambda} \not\subset A_{P_{w\mu}}$ which means that there exists a $\rho \in \Phi$ such that $\langle \lambda,\rho\rangle < 0$, $\langle w\mu,\rho\rangle \geq 0$. I.e. as above $\dim X_\lambda - \dim(X_\lambda \cap wX_\mu) \geq \frac{\|\chi\|r}{m} - 2$ which is bigger than $3\dim G \geq \dim B$ if $\|\chi\| \geq M$. This contradicts (7.7).

The case $A_{P_\mu} \not\subset A_{P_\lambda}$ is similar.

Theorem 7.2.4 and Theorem 7.2.7 lead to the following corollary.

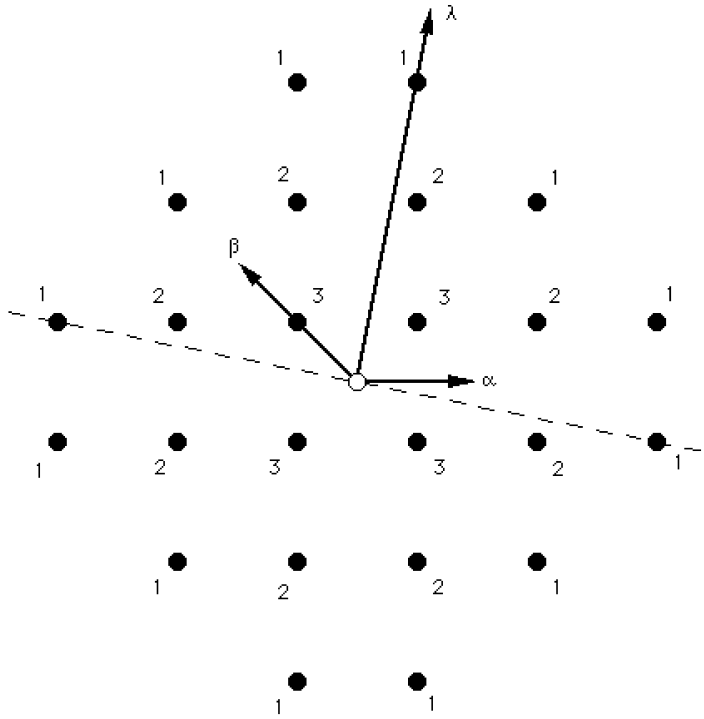
Corollary 7.2.8. *If G is simple of adjoint type then there are only a finite number of W such (*) is not satisfied.*

Proof . For a group of adjoint type all representations have their weights in the root lattice. Suppose that W is such that condition (*) does not hold. By Theorems 7.2.4 and 7.2.7 the irreducible subrepresentations of W have both their multiplicities and their highest weights bounded. Hence there are only a finite number of possibilities for W.

For irreducible representations, not having their highest weight in the root lattice, there is in general no boundedness result such as in Theorem 7.2.7. E.g. consider the following example :

Example 7.2.9. Let G be the simply connected group of type B_2 and let the simple roots be α and β such that $\|\beta\| > \|\alpha\|$. Furthermore let W be the representation with highest weight $3\alpha + \frac{1}{2}\beta$.

Then the weights of W, together with their multiplicities, are as follows.



Identify $X(T)_{\mathbb{R}}$ with $Y(T)_{\mathbb{R}}$ using (,), let λ be as indicated in the diagram and let w be the reflection corresponding to α.

Then $\dim X_\lambda - \dim(X_\lambda \cap wX_\lambda) = 1$, and since $X_\lambda \cap wX_\lambda$ is not B-invariant (the weights are not stable under adding the roots of B), $B(X_\lambda \cap wX_\lambda)$ is dense in X_λ and hence $G(X_\lambda \cap wX_\lambda)$ is dense in GX_λ. This implies by lemma 7.2.2 that $G \times^B X_\lambda \to GX_\lambda$ is not even birational.

It is clear that this example may be generalized to yield arbitrary big irreducible representations such that (*) does not hold. Similar examples may be constructed for other classical groups.

7.3. **Calculation of the character of $\mathcal{L}(GX_\lambda, X)$.** To apply Theorem 2.1 effectively, we need to know the G-structure on $\mathcal{L}(GX_\lambda, X)$. Throughout this subsection we assume that (*) holds. Assume $\lambda \in \Lambda$.

We will use the following diagram

$$\begin{array}{ccc} G\times^{P_\lambda} X_\lambda & \xrightarrow{i} & G\times^{P_\lambda} X \\ \downarrow & & \downarrow{\scriptstyle \pi_{P_\lambda,G}} \\ GX_\lambda & \xrightarrow{j} & X \end{array}$$

Lemma 7.3.1.

$$(\pi_{P_\lambda,G} i)_+ \mathcal{O}_{G\times^{P_\lambda} X_\lambda} \cong \mathcal{L}(GX_\lambda, X) \tag{7.9}$$

in $(G,\mathcal{D}_X)$*-qch.*

Proof . Since $G\times^{P_\lambda} X_\lambda \to GX_\lambda$ is a small resolution by hypothesis, it follows from the Riemann-Hilbert correspondence and [15, §6.2] that

$$(\pi_{P_\lambda,G} i)_+ \mathcal{O}_{G\times^{P_\lambda} X_\lambda} \cong \mathcal{L}(GX_\lambda, X) \tag{7.10}$$

(7.9) now follows by Proposition 3.1.2.

Hence we have to compute the G-character of $(\pi_{P_\lambda,G} i)_+ \mathcal{O}_{G\times^{P_\lambda} X_\lambda}$. (There is a slight abuse of terminology here since literally $(\pi_{P_\lambda,G} i)_+ \mathcal{O}_{G\times^{P_\lambda} X_\lambda}$ is a $\mathcal{O}_X$-module, but we consider it as an $R = SW$-module.)

We now use the following diagram

$$\begin{array}{ccc} G\times^{P_\lambda} X & \longrightarrow & G/P_\lambda \\ \downarrow{\scriptstyle \pi_{P_\lambda,G}} & & \downarrow{\scriptstyle \pi_{P_\lambda,G}} \\ X & \xrightarrow{f} & \operatorname{Spec}\mathbb{C} \end{array}$$

Taking the fiber over $[P_\lambda] \in G/P_\lambda$ induces an equivalence between $(G,\mathcal{O}_{G/P_\lambda})$-qch and the category of rational P_λ-representations. Below we denote the inverse of this functor by $\tilde{}$.

Since $i_+\mathcal{O}_{G\times^{P_\lambda} X_\lambda} = \mathcal{H}^{e_\lambda}_{G\times^{P_\lambda} X_\lambda}(G\times^{P_\lambda} X_\lambda, \mathcal{O}_{G\times^{P_\lambda} X})$, we obtain

$$f_*(\pi_{P_\lambda,G} i)_+(\mathcal{O}_{G\times^{P_\lambda} X_\lambda}) = \pi_{P_\lambda,G+}(H^{e_\lambda}_{X_\lambda}(X,\mathcal{O}_X)\tilde{}) \tag{7.11}$$

Here we consider $H^{e_\lambda}_{X_\lambda}(X,\mathcal{O}_X)$ as a graded (rational) P_λ-representation, equipped with its natural grading, and hence $H^{e_\lambda}_{X_\lambda}(X,\mathcal{O}_X)\tilde{}$ is a $\mathcal{O}_{G/P_\lambda}$-module.

Now we have to introduce some notation. If M is an additive monoid then we denote by $\mathbb{Z}[M]$ the "monoid ring" of M. I.e. the elements of $\mathbb{Z}[M]$ are given by

$$\sum_{m\in M} a_m[m] \qquad \text{(finite sum)} \tag{7.12}$$

with $[m][m'] = [m+m']$. By $\mathbb{Z}\{M\}$ we denote the abelian group of sums of the form (7.12), except that we do not require the sums to be finite. $\mathbb{Z}\{M\}$ is in an obvious way a $\mathbb{Z}[M]$-module, but it is not a ring. Provided that one is careful, elements of $\mathbb{Z}\{M\}$ may sometimes be interpreted as fractions over $\mathbb{Z}[M]$. See [8, §1] for a more precise statement.

Below we will use the notation e^m for $[m]$.

We will also need $\mathbb{Z}[\mathbb{Z}\oplus M]$ and $\mathbb{Z}\{\mathbb{Z}\oplus M\}$. In that case, for t a variable, we will put $[n\oplus m] = t^n e^m$ and we will use the more traditional notations $\mathbb{Z}[t][M]$ and $\mathbb{Z}\{t\}\{M\}$.

Put $\mathbf{P} = X(T)_{\mathbb{R}}$ and let $\mathbf{P}^{++}$ be the dominant weights in $\mathbf{P}$ with respect to B. If $\chi \in \mathbf{P}^{++}$ lies in the weight lattice then we denote by $V(\chi)$ the corresponding irreducible G-module.

Definition 7.3.2. A rational T-representation is (T-)bounded if its irreducible components occur with finite multiplicity.

If V is bounded then

$$[V]_T = \sum_{\chi \in X(T)} \operatorname{mult}_\chi V.e^\chi$$

defines an element of $\mathbb{Z}\{\mathbf{P}\}$. Similarly, if V is in addition a rational G-module then

$$[V]_G = \sum_{\substack{\chi \in \mathbf{P}^{++} \cap \\ \text{weight lattice}}} \operatorname{mult}_{V(\chi)} V.e^\chi$$

defines an element of $\mathbb{Z}\{\mathbf{P}^{++}\}$.

If $V = \oplus_{n\in\mathbb{Z}} V_n$ is a graded rational T-representation such that each V_n is bounded then

$$\mathcal{H}_T(V,t) = \sum_{t\in\mathbb{Z}} [V_n]_T t^n e^\chi \in \mathbb{Z}\{t\}\{\mathbf{P}\}$$

is the T-equivariant Hilbert series. The G-equivariant Hilbert series is defined in the same way, and defines an element of $\mathbb{Z}\{t\}\{\mathbf{P}^{++}\}$.

We will also consider the projection $p : \mathbf{P} \to \mathbf{P}^{++}$ where

$$p(z) = \begin{cases} z & \text{if } z \in \mathbf{P}^{++} \\ 0 & \text{otherwise} \end{cases}$$

and we extend p to maps $\mathbb{Z}\{\mathbf{P}\} \to \mathbb{Z}\{\mathbf{P}^{++}\}$, $\mathbb{Z}\{t\}\{\mathbf{P}\} \to \mathbb{Z}\{t\}\{\mathbf{P}^{++}\}$ which we will also denote by p.

Example 7.3.3. If $\lambda \in X(T)_{\mathbb{R}}$ then the homogeneous components of $H^{e_\lambda}_{X_\lambda}(X, \mathcal{O}_X)$ are bounded T-representations (this follows from [29]). Note that this is false in general if we replace X_λ by an arbitrary T-invariant linear subspace of X.

Definition 7.3.4. Let $Q \in \mathcal{Q}$, and let $\mathcal{M}$ be a G-equivariant quasi-coherent $\mathcal{O}_{G/Q}$-module. Then we say that $\mathcal{M}$ is bounded if the fiber $\mathcal{M}_{[Q]}$ (which is a rational Q-representation) is T-bounded.

If $\mathcal{M}$ is bounded then for $i \in \mathbb{N}$ the $[H^i(G/Q, \mathcal{M})]_G$ are defined. More generally, bounded modules are stable under inverse images, higher direct images and, in short, all other constructions we use below. We leave it to the reader to check this.

Now let $Q \in \mathcal{Q}$ and consider the following maps

$$\begin{tikzcd} G/B \arrow[r, "\pi_{B,Q}"] \arrow[d, "\pi_{B,G}"] & G/Q \arrow[d, "\pi_{Q,G}"] \\ \operatorname{Spec}\mathbb{C} \arrow[r, equal] & \operatorname{Spec}\mathbb{C} \end{tikzcd}$$

Lemma 7.3.5. *Let $\mathcal{M} \in (G, \mathcal{D}_{G/Q})$-wqch and assume that $\mathcal{M}$ is bounded. Then*

$$\sum_i (-)^i [R^i \pi_{Q,G+} \mathcal{M}]_G = \frac{(-)^{\dim Q/B}}{|\mathcal{W}_Q|} \sum_i (-)^i [R^i \pi_{B,G+} \pi^*_{Q,B} \mathcal{M})]_G$$

where $\mathcal{W}_Q$ is the Weyl group of Q.

Proof . This is basically the Leray spectral sequence, which yields

(7.13)
$$\begin{aligned}\sum_i (-)^i [R^i \pi_{B,G+} \pi^*_{Q,B} \mathcal{M}]_G &= \sum_{i,j} (-)^{i+j} [R^i \pi_{Q,G+} (R^j \pi_{B,Q+} \pi^*_{Q,B} \mathcal{M})]_G \\ &= \sum_{i,j} (-)^{i+j} [R^i \pi_{Q,G+} (\mathcal{M} \otimes_{\mathcal{O}_{G/Q}} R^j \pi_{B,Q+} \mathcal{O}_{G/B})]_G\end{aligned}$$

It now follows from the Riemann-Hilbert correspondence [5], [4] and the fact that flag varieties are simply connected

$$R^{j-\dim Q/B} \pi_{B,Q+} \mathcal{O}_{G/B} = \begin{cases} \mathcal{O}_{G/Q}^{\oplus |\{w \in \mathcal{W}_G | l(w) = j/2\}|} & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd} \end{cases} \tag{7.14}$$

The action of G on the righthand side of (7.14) is the obvious one. This follows from Proposition 3.1.2

Hence substitution of (7.14) in the righthand side of (7.13) gives

$$\sum_{i,j} (-)^{i+j} [R^i \pi_{Q,G+} (\mathcal{M} \otimes_{\mathcal{O}_{G/Q}} R^j \pi_{B,Q+} \mathcal{O}_{G/B}]_G = (-)^{\dim Q/B} |\mathcal{W}_Q| \sum_i (-)^i [R^i \pi_{Q,G+} \mathcal{M})]_G$$

Combining this with (7.13) gives what we want.

Now let $\overline{\rho}$ be half the sum of the positive roots of G.

Lemma 7.3.6. *Let V be a rational B-representation, bounded as T-representation. Then*

$$\sum_i (-)^i [H^i(G/B, \tilde{V})]_G = p \left(\sum_{w \in \mathcal{W}_G} (-)^{l(w)} e^{w\overline{\rho} - \overline{\rho}} w[V]_T \right) \tag{7.15}$$

Proof . Since the action of B on V is locally finite, and by additivity of Euler characteristic, we may assume that V is one-dimensional, i.e. a character χ of T. In that case (7.15) follows directly from Bott's theorem.

We are now ready to prove the principal result of this section.

Theorem 7.3.7. *Assume that condition (*) holds. Consider $\mathcal{L}(GX_\lambda, X)$ as a graded R-module. Then*

(7.16)
$$\mathcal{H}_G(\mathcal{L}(GX_\lambda, X), t) = (-)^{\dim G/P_\lambda} p \left(\left(\sum (-)^{l(w)} e^{w\overline{\rho} - \overline{\rho}} \right) \sum_{w \in \mathcal{W}_G / \mathcal{W}_\lambda} w \mathcal{H}_T (H^{d_\lambda}_{X_\lambda}(X, \mathcal{O}_X), t) \right)$$

Proof . Using lemma 7.3.1, (7.11) and lemma 7.3.5 it suffices to compute

$$\sum_i (-)^i \mathcal{H}_G (R^i \pi_{B,G+} (H^{e_\lambda}_{X_\lambda}(X, \mathcal{O}_X))\tilde{}, t) \tag{7.17}$$

where we now consider $H^{e_\lambda}_{X_\lambda}(X, \mathcal{O}_X)$ as a rational B-representation. We have to remember that we have dropped a factor $\frac{(-)^{\dim P_\lambda / B}}{|\mathcal{W}_\lambda|}$ in (7.17).

Now by the formula for direct images for projections in [5, VI, 5.3.1]

$$R^i \pi_{B,G+} (H^{e_\lambda}_{X_\lambda}(X, \mathcal{O}_X)\tilde{}) = \mathbb{H}^{i+\dim G/B} (\Omega^{\cdot}_{G/B} \otimes_{\mathcal{O}_{G/B}} H^{e_\lambda}_{X_\lambda}(X, \mathcal{O}_X)\tilde{})$$

Hence by an Euler characteristic type argument

(7.18)

$$\begin{aligned}
(7.17) &= \sum_{i,j}(-)^{i+j+\dim G/B}\mathcal{H}_G(H^i(G/B,\Omega^j_{G/B}\otimes_{\mathcal{O}_{G/B}} H^{e_\lambda}_{X_\lambda}(X,\mathcal{O}_X)\tilde{\ }),t)\\
&= \sum_{i,j}(-)^{i+j+\dim G/B}\mathcal{H}_G(H^i(G/B,(\Lambda^j(\mathfrak{g}/\mathfrak{b})^*\otimes_{\mathbb{C}} H^{e_\lambda}_{X_\lambda}(X,\mathcal{O}_X))\tilde{\ }),t)\\
&= p\left(\sum_{j,w}(-)^{j+l(w)+\dim G/B}e^{w\overline{\rho}-\overline{\rho}}w[\Lambda^j(\mathfrak{g}/\mathfrak{b})^*]_T w\mathcal{H}_T(H^{e_\lambda}_{X_\lambda}(X,\mathcal{O}_X),t)\right)
\end{aligned}$$

Now

$$\begin{aligned}
\sum_j(-)^j[\Lambda^j(\mathfrak{g}/\mathfrak{b})^*]_T &= \prod_{\rho\in\Phi^+}(1-e^{-\rho})\\
&= e^{-\overline{\rho}}\prod_{\rho\in\Phi^+}(e^{\rho/2}-e^{-\rho/2})\\
&= e^{-\overline{\rho}}\sum_{w\in\mathcal{W}_G}(-)^{l(w)}e^{w\overline{\rho}}
\end{aligned}$$

Substituting this in (7.18) yields

$$p\left((-)^{\dim G/B}(\sum_w(-)^{l(w)}e^{w\overline{\rho}-\overline{\rho}})\sum_{w\in\mathcal{W}_G}w\mathcal{H}_T(H^{e_\lambda}_{X_\lambda}(X,\mathcal{O}_X),t)\right)$$

Reintroducing the dropped factor $\frac{(-)^{\dim P_\lambda/B}}{|\mathcal{W}_\lambda|}$ yields (7.16). (We have used that for $w\in\mathcal{W}_\lambda$, $H^{e_\lambda}_{X_\lambda}(X,\mathcal{O}_X)$ is w-stable.)

7.4. **Some examples.** Below we will discuss some applications and examples of Theorems 2.1 and 7.3.7. If a reductive group G acts on a variety Y then we say that $y\in Y$ is G-stable if y has closed orbit and finite stabilizer.

Example 7.4.1. Here we compute the contribution of the term in (2.1) corresponding to $\lambda=0$. We assume that X has a G-stable point.

First we have to identify those $w\in\mathcal{W}_G$ for which $(w,0)$ is admissible. Clearly $(A_B)_0=0$ and $0\in A_B^{(w,\lambda)} \iff S\cap w^{-1}\Phi^+=\emptyset$ (lemma 6.8). This will happen only if $w=w_l$, the longest element in $\mathcal{W}_G$. Hence only $(w_l,0)$ is admissible and $\Psi_{w_l,0}=\emptyset$, or

$$\tilde{H}^i(\Psi_{w_l,0},\mathbb{C})=\begin{cases}\mathbb{C} & \text{if } i=-1\\ 0 & \text{otherwise}\end{cases}$$

Therefore, if (*) holds, $\lambda=0$ will contribute $H^{\dim X}_{\{0\}}(X,\mathcal{O}_X)=H^{\dim W}_{R^+}(R)$ to $H^n_{X^u}(X,\mathcal{O}_X)$ where $n+\dim T-\dim X+2l(w_l)-1=-1$ or $n=\dim X-\dim G=\dim X/\!\!/G=\dim R^G$.

Example 7.4.2. Now let $G=\mathrm{Sl}_2$, i.e. $G=\mathrm{Sl}(V)$, $\dim V=2$. In that case (*) holds, unless W is, up to trivial representations, equal to V or S^2V, i.e. if and only if $W=W^*$ has no G-stable point.

Assume now that (*) does indeed hold. We may identify $X(T)_{\mathbb{R}}\cong\mathbb{R}\cong Y(T)_{\mathbb{R}}$ such that $\langle\ ,\ \rangle$ is multiplication. Let ω be the fundamental weight of G. I.e. $V=V(\omega)$. We will assume that ω is identified with $+1$ in $\mathbb{R}$. Using our identification of $X(T)_{\mathbb{R}}$ and $Y(T)_{\mathbb{R}}$ we may clearly assume that $\Lambda = \{0, -\omega\}$ and we have to find the admissible pairs (w, λ) in $\mathcal{W}_G \times \Lambda$. The only case not covered by example 7.4.1 is that of $(\mathrm{id}, -\omega)$.

Now $(A_B)_{-\omega} = \mathbb{R}_{<0}$, $\Phi_{-\omega} = \emptyset$ and hence by lemma 6.8, $A_B^{(\mathrm{id},-\omega)} = A_B = \mathbb{R}_{\le 0}$. Hence $\Psi_{\mathrm{id},-\omega} = \{0, -1\}$, i.e. a set of two points. Consequently

$$\tilde{H}^i(\Psi_{\mathrm{id},-\omega}, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

and we obtain a contribution $\mathcal{L}(GX_{-\omega}, X)$ to $H^n_{X^u}(X, \mathcal{O}_X)$ where $n + \dim T - f_{-\omega} + 2l(\mathrm{id}) - 1 = 0$ or $n = f_{-\omega}$. I.e. $H^n(X, \mathcal{O}_X)$ will only be non-zero if $n = \dim X /\!\!/ G$ or if $n = f_{-\omega} = \mathrm{codim}(GX_{-\omega}, X)$.

$\mathcal{H}_G(\mathcal{L}(GX_\lambda, X), t)$ may be computed by (7.16). We obtain

(7.19)
$$\mathcal{H}_G(\mathcal{L}(GX_\lambda, X), t) = p\left((e^{-2\omega} - 1)\left(\mathcal{H}_T(H^{e_{-\omega}}_{X_{-\omega}}(X, \mathcal{O}_X), t) + \mathcal{H}_T(H^{e_\omega}_{X_\omega}(X, \mathcal{O}_X), t)\right)\right)$$

However from the description of the weights of $H^{d_\lambda}_{X_\lambda}(X, \mathcal{O}_X)$ given in [29] one easily sees that (7.19) simplifies to

$$\mathcal{H}_G(\mathcal{L}(GX_\lambda, X), t) = (e^{-2\omega} - 1)\mathcal{H}_T(H^{e_{-\omega}}_{X_{-\omega}}(X, \mathcal{O}_X), t)$$

Hence again using [29], we recover the results of [26] and [9].

Example 7.4.3. Now we assume $G = SL(V)$, $\dim V = 3$. Here it is impossible to treat every W, since, unlike in the case of Sl_2, each representation is essentially different. Therefore, we restrict ourselves to a particular case, namely $W = V^m$. It is easily verified that condition (*) holds if $m \ge 3$, which we assume.

Choose B and T. We may identify $X(T)_{\mathbb{R}} \cong \mathbb{R}^2 \cong Y(T)_{\mathbb{R}}$ in a $\mathcal{W}_G$-equivariant way, with the additional property that $\langle\ ,\ \rangle$ becomes the ordinary scalar product on $\mathbb{R}^2$.

Let $\omega_{1,2}$ be the fundamental weights of G and assume that $V(\omega_1) = V$. Via the above identification, we consider $\omega_{1,2}$ also as elements of $Y(T)_{\mathbb{R}}$. Then it is easy to see that we may take for $\Lambda : \{0, -\omega_1, -\omega_2\}$. Let $s_1, s_2 \in \mathcal{W}_G$ be the fundamental reflections on $X(T)_{\mathbb{R}}$, which fix respectively ω_1 and ω_2. Then $\mathcal{W}_{-\omega_1} = \{\mathrm{id}, s_1\}$, $\mathcal{W}_{-\omega_2} = \{\mathrm{id}, s_2\}$.

Again we have to determine the admissible pairs. Excluding $\lambda = 0$, which was covered by example 7.4.1, there are 4 possibilities to consider : $(\mathrm{id}, -\omega_1)$, $(s_1, -\omega_1)$, $(\mathrm{id}, -\omega_2)$ and $(s_2, -\omega_2)$. A straightforward computation shows that these are all admissible, but $\Psi_{\mathrm{id},-\omega_1}$, $\Psi_{s_2,-\omega_2}$ are acyclic. On the other hand, $\Psi_{s_1,-\omega_1}$ is homotopic to a set of two points, whereas $\Psi_{\mathrm{id},-\omega_2}$ is homeomorphic to a circle.

Hence we will have a contribution $\mathcal{L}(GX_{-\omega_1}, X)$ in $H^n_{X^u}(X, \mathcal{O}_X)$ where $n + \dim T - f_{-\omega_1} + 2l(s_1) - 1 = 0$ or $n = f_{-\omega_1} - 3 = 2m - 5$ and a contribution $\mathcal{L}(GX_{-\omega_2}, X)$ in $H^n_{X^u}(X, \mathcal{O}_X)$ where $n + \dim T - f_{-\omega_2} + 2l(\mathrm{id}) - 1 = 1$ or $n = f_{-\omega_2} = m - 2$.

One noteworthy feature of this example is that although $X^u = GX_{-\omega_1}$, $GX_{-\omega_2}$ plays a role in the description of $H^*_{X^u}(X, \mathcal{O})$. I.e. not only the irreducible components of X^u count (as one perhaps, very naively, could hope for).

To complete the description of $H^i_{X^u}(X, \mathcal{O}_X)$ we have to determine the characters of $\mathcal{L}(GX_{-\omega_1}, X)$ and $\mathcal{L}(GX_{-\omega_2}, X)$. This we do next using (7.16) and the descriptions of the weights of $H_{X_\lambda}(X, \mathcal{O}_X)$ given in [29]. Unfortunately the computations are somewhat complicated and we needed a computer to obtain the following results.

$$\mathcal{H}_G(\mathcal{L}(GX_{-\omega_1}, X), t) = p\left(\sum_{\substack{a\geq -2, b\geq -1\\ c\geq -2}} P_1(a,b,c)e^{(b+c+m)\omega_1+(a-b)\omega_2}t^{-a-b+c-2m}\right)$$

where

$$P_1(a,b,c) = \frac{1}{(m-1)(m-2)^2}(a+c+m+2)(a-b+1)(c+b+m+1)\times \binom{a+m-1}{m-3}\binom{b+m-2}{m-3}\binom{c+m-1}{m-3} \tag{7.20}$$

and

$$\mathcal{H}_G(\mathcal{L}(GX_{-\omega_2}, X), t) = p\left(\sum_{\substack{a\geq -2, b\geq -1\\ c\geq -2}} P_2(a,b,c)e^{(c-b)\omega_1+(m+a+b)\omega_2}t^{-a+b+c-m}\right)$$

where

$$P_2(a,b,c) = \frac{1}{(m-1)(m-2)^2}(a+c+m+2)(c-b+1)(a+b+m+1)\times \binom{a+m-1}{m-3}\binom{b+m-2}{m-3}\binom{c+m-1}{m-3} \tag{7.21}$$

Hence the representations that occur in $\mathcal{L}(GX_{-\omega_1}, X)$ will have highest weights of the form $x\omega_1 + y\omega_2$ where $x = b + c + m$, $y = a - b$ with the properties $a \geq -2$, $b \geq -1$, $c \geq -2$, $a+c+m+2 \neq 0$, $a-b+1 \neq 0$, $c+b+m+1 \neq 0$, $b+c+m \geq 0$, $a - b \geq 0$.

Of course, these conditions are highly redundant. A minimal subset is given by $b \geq -1$, $c \geq -2$, $a \geq b$ which gives the constraints $x \geq m-3$ and $y \geq 0$.

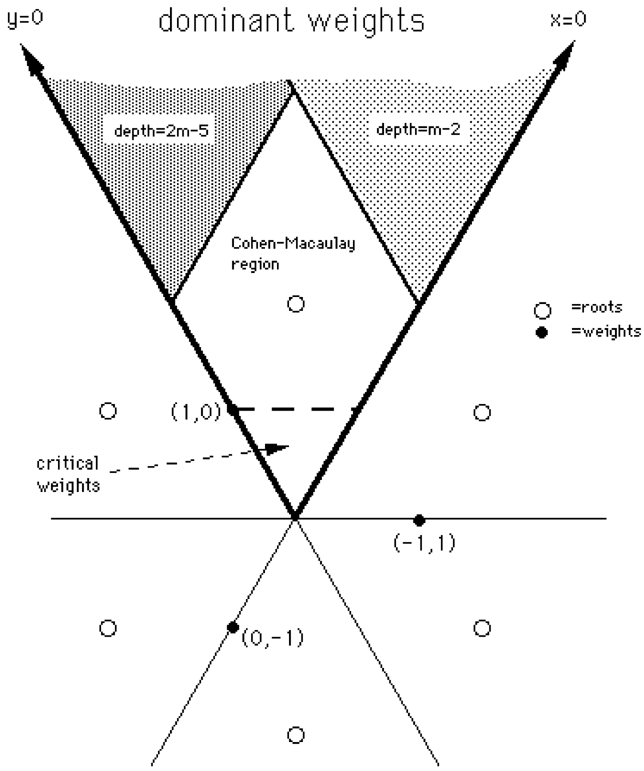
A similar computation shows that the representations in $\mathcal{L}(GX_{-\omega_2}, X)$ have highest weights of the form $x\omega_1 + y\omega_2$ where this time $x \geq 0$, $y \geq m-3$.

We may now summarize our results as follows. Let χ be a character of G with corresponding highest weight $x\omega_1 + y\omega_2$, $x \geq 0$, $y \geq 0$. Then

$$\operatorname{depth} R^G_\chi = \begin{cases} m-2 & \text{if } y \geq m-3 \\ 2m-5 & \text{if } x \geq m-3,\ y < m-3 \\ 3m-8 & \text{if } x < m-3,\ y < m-3 \end{cases}$$

Now we recall that Stanley's criterion [24] says that R^G_χ is Cohen-Macaulay if χ is "critical". This conjecture was almost completely proved in [28]. Using [28, Prop. 1.4] it is easily seen that χ is critical for (G, W) if $x + y + 4 < m$.

The results in this example may be summarized in the following figure (which is for $m = 5$).



I.e. we see that, in contrast with the case $G = \mathrm{Sl}_2$, Stanley's criterion is not very precise.

Appendix A. A theorem about $\mathcal{D}$-modules

In this appendix we prove a theorem about $\mathcal{D}$-modules, which is a generalization of [2, 4.2.5,4.2.6]. It is presumably well-known but I have been unable to locate a reference. As usual we let the base field be $\mathbb{C}$. $\pi : Y \to X$ will be a smooth map of smooth quasi-projective varieties over $\mathbb{C}$.

Theorem A.1. *Assume that the fibers of π are non-empty and connected of constant dimension d. Then*

1. *The functor $\pi^* : \mathcal{D}_X$-qch $\to \mathcal{D}_Y$-qch is fully faithful.*
2. *Suppose $\mathcal{M} \in \mathcal{D}_X$-qch and $\mathcal{N}' \subset \pi^*\mathcal{M}$ in $\mathcal{D}$-qch. Then there exists (a unique) $\mathcal{N} \subset \mathcal{M}$ in $\mathcal{D}_X$-qch such that $\pi^*\mathcal{N} = \mathcal{N}'$.*

Proof . Let $\mathcal{M} \in \mathcal{D}_X$-qch and let $\Omega^{\cdot}_{Y/X}(\mathcal{M})$ be the relative De Rham complex. Then using the fact that π is locally the product of an etale map and a projection, one shows that $\pi_* H^0(\Omega^{\cdot}_{Y/X}(\pi^*\mathcal{M}))$ carries a $\mathcal{D}_X$-module structure, and the canonical map

$$\mathcal{M} \to \pi_* H^0(\Omega^{\cdot}_{Y/X}(\pi^*\mathcal{M})) \tag{A.1}$$

is $\mathcal{D}_X$-linear (of course this is entirely classical). (A.1) is the map which for $U \subset X$ open identifies the elements of $\mathcal{M}(U)$ with the relative horizontal sections of $\pi^*\mathcal{M}$ on $\pi^{-1}(U)$.

We claim that (A.1) is an isomorphism. This easily implies (1) since then $\pi^*\mathcal{M}$ is generated by its relative horizontal sections, and a $\mathcal{D}_Y$-linear map must respect these.

Our claim does not depend on the $\mathcal{D}_X$-modules structure of $\mathcal{M}$, so we may as well assume that $\mathcal{M} \in \mathcal{O}_X$-qch. Since $\mathcal{M}$ is the direct limit of coherent $\mathcal{O}_X$-modules, we may furthermore assume that $\mathcal{M}$ is coherent.

Our situation is local for the etale topology on X so we may assume that π has a section e. Restricting to e yields a retraction of (A.1) and therefore (A.1) is injective.

To prove surjectivity we have to show that if two relative horizontal sections of $\pi^*\mathcal{M}$ are equal on e then they are equal everywhere. Suppose that this is not the case. By taking differences we may assume that we have a non-zero relative horizontal section f of $\pi_*\mathcal{M}$, which is zero on e.

Assume that $\mathcal{N}$ is a submodule of $\mathcal{M}$. Then there is an exact sequence

$$0 \to \pi_* H^0(\Omega^{\cdot}_{Y/X}(\pi^*\mathcal{N})) \to \pi_* H^0(\Omega^{\cdot}_{Y/X}(\pi^*\mathcal{M})) \to \pi_* H^0(\Omega^{\cdot}_{Y/X}(\pi^*(\mathcal{M}/\mathcal{N})))$$

This shows that f either has non-zero image in $\pi_* H^0(\Omega^{\cdot}_{Y/X}(\pi^*(\mathcal{M}/\mathcal{N})))$, or lies in $\pi_* H^0(\Omega^{\cdot}_{Y/X}(\pi^*\mathcal{N}))$. By repeatedly applying this, and by shrinking X, we may assume that X is irreducible and that $\mathcal{M}$ is a torsion free $\mathcal{O}_X$-module. But then $\mathcal{M}$ injects in the localization at the generic point of X. Hence we may assume that $X = \operatorname{Spec} F$, with F a field. Then $\mathcal{M}$ is a finite dimensional vector space over F and hence we may assume that $\mathcal{M}$ is one-dimensional, that is $\pi^*\mathcal{M} = \mathcal{O}_Y$. Since Y is connected the horizontal sections of $\mathcal{O}_Y$ are the contstants, and hence they cannot be zero on e.

To prove (2) let $\mathcal{M}$, $\mathcal{N}'$ be as in the statement of (2). $\mathcal{N}$, if it exists is unique because of the faithfulness of π. We put $\mathcal{N} = \pi_* H^0(\Omega_{Y/X}(\mathcal{N}')) \hookrightarrow \pi_* H^0(\Omega_{Y/X}(\pi^*\mathcal{M})) = \mathcal{M}$, and we claim that the natural map $\pi^*\mathcal{N} \to \mathcal{N}'$ is an isomorphism.

Again this claim does not refer to the $\mathcal{D}_X$-module structure on $\mathcal{M}$ and we may therefore assume that $\mathcal{M}$ is a quasicoherent $\mathcal{O}_X$-module, and $\mathcal{N}' \subset \pi^*\mathcal{M}$ a quasicoherent $\mathcal{D}_{Y/X}$-module ($\mathcal{D}_{Y/X}$ is the sheaf of algebras, generated by $\mathcal{O}_Y$ and $\mathcal{T}_{Y/X}$).

Since $\mathcal{M}$ is a union of coherent $\mathcal{O}_X$-modules, we may furthermore assume that $\mathcal{M}$ itself is coherent, which is what we will do.

Assume first that $X = \operatorname{Spec} F$, F a field. Then $\mathcal{M}$ is a finite dimensional vector space and hence $\pi^*\mathcal{M} = \mathcal{O}_Y^{\oplus n}$ for some n. Now $\mathcal{O}_Y$ is a simple $\mathcal{D}_Y$-module, and hence $\mathcal{N}' = \mathcal{O}_Y^{\oplus m}$ for some $m \leq n$. This proves our claim in this special case.

Let X now be arbitrary again. We use the following observation. Suppose there is an exact sequence on X

$$0 \to \mathcal{M}_1 \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{M}_2 \to 0 \tag{A.2}$$

and an exact diagram, where the vertical arrow are inclusions

$$\begin{tikzcd}
0 \arrow[r] & \pi^*\mathcal{M}_1 \arrow[r, "\pi^*\alpha"] & \pi^*\mathcal{M} \arrow[r, "\pi^*\beta"] & \pi^*\mathcal{M}_2 \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{N}'_1 \arrow[r] \arrow[u] & \mathcal{N}' \arrow[r] \arrow[u] & \mathcal{N}'_2 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Then, if the claim is true for $\mathcal{N}'_1$ and $\mathcal{N}'_2$, it is also true for $\mathcal{N}'$.

To see this write $\mathcal{N}'_1 = \pi^*\mathcal{N}_1$, $\mathcal{N}'_2 = \pi^*\mathcal{N}_2$ with $\mathcal{N}_1 \subset \mathcal{M}_1$, $\mathcal{N}_2 \subset \mathcal{M}_2$.

We construct a new exact diagram as follows

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & \mathcal{V} \arrow[r, "\cong"] \arrow[u] & \mathcal{U} \arrow[u] & \\
0 \arrow[r] & \pi^*\mathcal{N}_1 \arrow[r] & \pi^*\mathcal{M} \arrow[r] \arrow[u] & \pi^*\mathcal{M}_3 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \pi^*\mathcal{N}_1 \arrow[r] \arrow[u, "\cong"] & \mathcal{N}' \arrow[r] \arrow[u] & \pi^*\mathcal{N}_2 \arrow[r] \arrow[u] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Here of course $\mathcal{M}_3 = \mathcal{M}/\mathcal{N}_1$. Now $\mathcal{U} = \pi^*\mathcal{M}_4$ where $\mathcal{M}_4 = \mathcal{M}_3/\mathcal{N}_2$. Then $\mathcal{N}' = \pi^* \ker(\mathcal{M} \to \mathcal{M}_4)$. This proves the observation.

Hence assume that we have a counter example to (2) where X is of minimal dimension. By using the above observation repeatedly and by shrinking X we may assume that X is irreducible and that $\mathcal{M}$ is torsion free of rank one.

Let η be the generic point of X. By our discussion for the case X a point, it follows that $\pi^*\mathcal{N}_\eta \to \mathcal{N}'_\eta$ is an isomorphism. If $\mathcal{N}' = 0$ then there is nothing to prove, so we assume that $\mathcal{N}' \neq 0$. Then $\mathcal{N}'_\eta \neq 0$ since $\pi^*\mathcal{M}$ contains no submodules with smaller support.

Hence $\mathcal{N} \neq 0$. But then $\mathcal{M}/\mathcal{N}$ has strictly smaller support than X and hence, by hypothesis, (2) is true for $\mathcal{N}'/\pi^*\mathcal{N}$. Then the following diagram shows that (2) is also true for $\mathcal{N}$, yielding a contradiction.

$$\begin{tikzcd}
0 \arrow[r] & \pi^*\mathcal{N} \arrow[r] & \pi^*\mathcal{M} \arrow[r] & \pi^*(\mathcal{M}/\mathcal{N}) \arrow[r] & 0 \\
0 \arrow[r] & \pi^*\mathcal{N} \arrow[r] \arrow[u, "\cong"] & \mathcal{N}' \arrow[r] \arrow[u] & \mathcal{N}'/\pi^*\mathcal{N} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

□

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