HOCHSCHILD COHOMOLOGY OF ABELIAN CATEGORIES AND RINGED SPACES

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Dedicated to Michael Artin on the occasion of his seventieth birthday.

ABSTRACT. This paper continues the development of the deformation theory of abelian categories introduced in a previous paper by the authors. We show first that the deformation theory of abelian categories is controlled by an obstruction theory in terms of a suitable notion of Hochschild cohomology for abelian categories. We then show that this Hochschild cohomology coincides with the one defined by Gerstenhaber, Schack and Swan in the case of module categories over diagrams and schemes and also with the Hochschild cohomology for exact categories introduced recently by Keller. In addition we show in complete generality that Hochschild cohomology satisfies a Mayer-Vietoris property and that for constantly ringed spaces it coincides with the cohomology of the structure sheaf.

CONTENTS

1. Introduction

In the rest of this paper k is an arbitrary commutative base ring but for simplicity we will assume in this introduction that k is a field.

Motivated by our work on the infinitesimal deformation theory of abelian categories [30] our aim in this paper is to develop a theory of Hochschild cohomology for abelian categories and ringed spaces. The corresponding theory for Hochschild (and cyclic) *homology* is by now well established [21, 26, 41]. The theory for Hochschild cohomology has a rather different flavour since it is less functorial, but nevertheless it still has good stability and agreement properties.

We start with the case of k-linear categories. The Hochschild complex $\mathbf{C}(\mathfrak{a})$ of a k-linear category $\mathfrak{a}$ is defined by

$$(1.1) \quad \mathbf{C}^p(\mathfrak{a}) = \prod_{A_0,\ldots,A_p \in \mathrm{Ob}(\mathfrak{a})} \mathrm{Hom}_k(\mathfrak{a}(A_{p-1}, A_p) \otimes_k \cdots \otimes_k \mathfrak{a}(A_0, A_1), \mathfrak{a}(A_0, A_p))$$

with the usual differential (see [31]). As in the case of associative algebras, the Hochschild complex of $\mathfrak{a}$ carries a considerable amount of "higher structure" containing in particular the classical cup-product and the Gerstenhaber bracket. This extra structure may be summarized conveniently by saying that $\mathbf{C}(\mathfrak{a})$ is an algebra over the so-called B_∞-operad [16, 19].

Let $\mathcal{A}$ be a k-linear abelian category. In this paper we define the *Hochschild complex* of $\mathcal{A}$ as

$$(1.2) \qquad\qquad \mathbf{C}_{\mathrm{ab}}(\mathcal{A}) = \mathbf{C}(\mathsf{InjInd}(\mathcal{A}))$$

where $\mathsf{Ind}\mathcal{A}$ is the abelian category of Ind-objects over $\mathcal{A}$ [3, Expose I] and $\mathsf{InjInd}(\mathcal{A})$ denotes the full subcategory of injective objects in $\mathsf{Ind}(\mathcal{A})$. It is understood here that $\mathsf{Ind}\mathcal{A}$ is computed with respect to a universe in which $\mathcal{A}$ is small. In the rest of this introduction, for the purpose of exposition, we will ignore such settheoretic complications (see §2.1,§2.5 below).

As indicated above the initial motivation behind (1.2) is the infinitesimal deformation theory of k-linear abelian categories developed in [30] (see §3 below). We prove (Theorem 3.1):

- the deformation theory of a k-linear abelian category $\mathcal{A}$ is controlled by an obstruction theory involving $H\mathbf{C}^2_{\mathrm{ab}}(\mathcal{A})$ and $H\mathbf{C}^3_{\mathrm{ab}}(\mathcal{A})$.

In §6 we prove basic results about the Hochschild cohomology of k-linear abelian categories. In particular we show (Theorem 6.6 and Corollaries 6.8,6.9):

- If $\mathcal{A}$ has itself enough injectives then $\mathbf{C}_{\mathrm{ab}}(\mathcal{A}) \cong \mathbf{C}(\mathsf{Inj}\mathcal{A}))$ (where here and below $\cong$ means the existence of an isomorphism in the homotopy category of B_∞-algebras). A dual statement holds of course if $\mathcal{A}$ has enough projectives.
- In general we have $\mathbf{C}_{\mathrm{ab}}(\mathcal{A}) \cong \mathbf{C}_{\mathrm{ab}}(\mathsf{Ind}\mathcal{A})$.
- If A is a k-algebra then the Hochschild cohomology of the abelian category $\mathsf{Mod}(A)$ coincides with the Hochschild cohomology of A.

We also show that the Hochschild complex of an abelian category $\mathcal{A}$ is the same as the Hochschild complexes of suitable DG-categories [20] associated to $\mathcal{A}$. The definition of the Hochschild complex of a DG-category is an obvious extension of (1.1) (see §2.4 below). Let $^e\mathcal{A}$ be the full DG-subcategory of $C(\mathsf{Ind}(\mathcal{A}))$ spanned by all positively graded complexes of injectives whose only cohomology is in degree zero and lies in $\mathcal{A}$ and let $^eD^b(\mathcal{A})$ be spanned by all left bounded complexes of injectives with bounded cohomology in $\mathcal{A}$. Then $^eD^b(\mathcal{A})$ is an exact DG-category such that $H^*(^eD^b(\mathcal{A}))$ is the graded category associated to $D^b(\mathcal{A})$. Hence $^eD^b(\mathcal{A})$ is a DG-"enhancement" [7, 8] for the triangulated category $D^b(\mathcal{A})$.

We prove (Theorem 6.1)

$$(1.3) \qquad \mathbf{C}_{\mathrm{ab}}(\mathcal{A}) \cong \mathbf{C}(^e\mathcal{A}) \cong \mathbf{C}(^eD^b(\mathcal{A}))$$

Using (1.3) we may construct a homomorphism of graded rings (see Prop. 4.5)

$$\sigma_\mathcal{A} : H\mathbf{C}^*_{\mathrm{ab}}(\mathcal{A}) \longrightarrow Z(D^b(\mathcal{A}))$$

where we view $D^b(\mathcal{A})$ as a graded category. The homogeneous elements of $Z(D^b(\mathcal{A}))$ consist of tuples $(\phi_M)_M$ with $M \in \mathrm{Ob}(D^b(\mathcal{A}))$ and $\phi_M \in \mathrm{Ext}^*_\mathcal{A}(M, M)$ satisfying a suitable compatibility condition (see §4.5). Thus we may think of $\sigma_\mathcal{A}$ as defining "universal" elements in $\mathrm{Ext}^*_\mathcal{A}(M, M)$ for every $M \in D^b(\mathcal{A})$. These universal elements are closely related to Atiyah classes in algebraic geometry. See for example [9].

The isomorphisms in (1.3) also connect $\mathbf{C}_{\mathrm{ab}}(\mathcal{A})$ to Keller's recent definition of the Hochschild complex of an exact category [19]. If $\mathcal{E}$ is an exact category then by definition $\mathbf{C}_{\mathrm{ex}}(\mathcal{E}) = \mathbf{C}(\mathcal{Q})$ for a DG-quotient [19, 10] $\mathcal{Q}$ of $Ac^b(\mathcal{E}) \longrightarrow C^b(\mathcal{E})$, where $C^b(\mathcal{E})$ is the DG-category of bounded complexes of $\mathcal{E}$-objects and $Ac^b(\mathcal{E})$ is its full DG-subcategory of acyclic complexes. If we equip an abelian category $\mathcal{A}$ with its canonical exact structure then $^eD^b(\mathcal{A})$ is a DG-quotient of $Ac^b(\mathcal{E}) \longrightarrow C^b(\mathcal{E})$ (see Lemma 6.3). Hence $\mathbf{C}_{\mathrm{ab}}(\mathcal{A}) \cong \mathbf{C}_{\mathrm{ex}}(\mathcal{A})$.

In §7 we specialize to ringed spaces. If $(X, \mathcal{O})$ is a ringed space then we define

$$\mathbf{C}(X) = \mathbf{C}_{\mathrm{ab}}(\mathsf{Mod}(X))$$

where $\mathsf{Mod}(X)$ is the category of sheaves of right $\mathcal{O}$-modules. Note that in this definition the bimodule structure of $\mathcal{O}$ does not enter explicitly. We show that $H\mathbf{C}^*(-)$ defines a "nice" cohomology theory for $(X, \mathcal{O})$ in the sense that it has the following properties (§7.4 and Theorem 7.9.1).

- $H\mathbf{C}^*(-)$ is a contravariant functor on open embeddings.

- Associated to an open covering $X = U \cup V$ there is a Mayer-Vietoris sequence

$$\cdots \to H\mathbf{C}^{i-1}(U \cap V) \to H\mathbf{C}^i(X) \to H\mathbf{C}^i(U) \oplus H\mathbf{C}^i(V) \to H\mathbf{C}^i(U \cap V) \to \cdots$$

- If $\mathcal{O}$ is the constant sheaf $\underline{k}$ with values in k then

(1.4) $$H\mathbf{C}^*(X, \underline{k}) \cong H^*(X, \underline{k})$$

where the righthandside is the usual derived functor cohomology of $\underline{k}$.

In [5] Baues shows that the singular cochain complex of a topological space is a B_∞-algebra. Thus (1.4) suggests that $\mathbf{C}(X, \mathbb{Z})$ should be viewed as an algebraic analog of the singular cochain complex of X.

We also show that under suitable conditions $H\mathbf{C}^*(X)$ coincides with the Hochschild cohomology theories for ringed spaces and schemes defined by Gerstenhaber, Schack and Swan. More precisely we show:

- Assume that X has a basis $\mathcal{B}$ of *acyclic opens*, i.e. for $U \in \mathcal{B}$: $H^i(U, \mathcal{O}_U) = 0$ for $i > 0$. Then $H\mathbf{C}^*(X)$ coincides with the Gerstenhaber-Schack cohomology [13, 14] of the restriction of $\mathcal{O}$ to $\mathcal{B}$, considered as a diagram over the partially ordered set $\mathcal{B}$.
- If X is a quasi-compact separated scheme then $H\mathbf{C}^*(X)$ coincides with the Hochschild cohomology for X as defined by Swan in [39].

We recall that for a smooth scheme the Hochschild complex defined by Swan is quasi-isomorphic to the one defined by Kontsevich [23] in terms of differential operators (see [39, 42]).

Now let X be a quasi-compact separated scheme and denote by $\mathsf{Qch}(X)$ the category of quasi-coherent sheaves on X. If X is in addition noetherian then let $\mathsf{coh}(X)$ be the coherent sheaves on X. We prove that there are isomorphisms (see Theorem 7.5.1 and Corollaries 7.7.2, 7.7.3)

(1.5) $$\mathbf{C}(X) \cong \mathbf{C}_{\mathrm{ab}}(\mathsf{Qch}(X)) \cong \mathbf{C}_{\mathrm{ab}}(\mathsf{coh}(X))$$

whenever the notations make sense.

The first isomorphism in (1.5) is proved by relating the Hochschild cohomology of X to that of a finite open affine covering of X. To be more precise let $X = A_1 \cup \cdots \cup A_n$ be such a covering and let $\mathcal{A}$ be the closure of $\{A_1, \ldots, A_n\}$ under intersections. Since X is separated, $\mathcal{A}$ consists of affine opens. We define a linear category $\mathfrak{a}$ with the same objects as $\mathcal{A}$ by putting

$$\mathfrak{a}(U, V) = \begin{cases} \Gamma(U, \mathcal{O}_U) & \text{if } U \subset V \\ 0 & \text{otherwise} \end{cases}$$

We construct an isomorphism (Corollary 7.7.2)

$$\mathbf{C}(X) \cong \mathbf{C}(\mathfrak{a})$$

In particular if $X = \operatorname{Spec} R$ is itself affine then $\mathbf{C}(X) \cong \mathbf{C}(R)$.

We are extremely grateful to Bernhard Keller for freely sharing with us many of his ideas and for making available the preprint [19]. While preparing the current manuscript the authors had independently discovered the main result of [19] (with the same proof) but nevertheless the presentation in [19] made it possible to clarify and generalize many of our original arguments.

The second author learned about the connection between Hochschild cohomology and Atiyah classes in an interesting discussion with Ragnar-Buchweitz at a sushi restaurant in Berkeley during the workshop on non-commutative algebraic geometry at MSRI in Februari 2000.

2. Preliminaries, conventions and notations

2.1. **Universes.** The results in this paper are most conveniently stated for small categories. However we sometimes need non-small categories as well, for example to pass from a category to its category of Ind-objects. Therefore we take the theory of universes as our set theoretical foundation since this basically allows us to assume that any category is small. For a brief introduction to the theory of universes and to some related terminology which we will use in this paper, we refer the reader to [30]. Our convention is that we fix a universe $\mathcal{U}$, and all terminology (small, Grothendieck, . . .) and all constructions (Ab, Mod, Ind, . . .) are implicitly prefixed by $\mathcal{U}$. Unless otherwise specified all categories will be $\mathcal{U}$-categories, i.e. their Hom-sets are $\mathcal{U}$-small. Individual objects such as rings and modules are also assumed to be $\mathcal{U}$-small.

2.2. **DG-categories.** We will assume that the reader has some familiarity with DG-categories and model categories. See for example [20, 18].

Throughout we fix a commutative ring k and we assume that all categories are k-linear. Unadorned tensor products and Hom's are over k. On first reading one may wish to assume that k is a field as it technically simplifies many definitions and proofs (see for example the next section).

Let $\mathfrak{a}$ be a DG-category. Associated to $\mathfrak{a}$ is the corresponding graded category (for which we use no separate notation), which is obtained by forgetting the differential and the categories $H^0(\mathfrak{a})$ and $H^*(\mathfrak{a})$ with $\mathrm{Ob}(H^0(\mathfrak{a})) = \mathrm{Ob}(H^*(\mathfrak{a})) = \mathrm{Ob}(\mathfrak{a})$ and

$$H^0(\mathfrak{a})(A, B) = H^0(\mathfrak{a}(A, B))$$
$$H^*(\mathfrak{a})(A, B) = H^*(\mathfrak{a}(A, B))$$

$H^0(\mathfrak{a})$ is sometimes referred to as the *homotopy category* of $\mathfrak{a}$.

Now assume that $\mathfrak{a}$ is small. We consider the DG-category

$$\mathsf{Dif}(\mathfrak{a}) = \mathsf{DGFun}(\mathfrak{a}^{\mathrm{op}}, C(k))$$

of (right) DG-modules. The derived category $D(\mathfrak{a})$ is the localized category

$$\mathsf{Dif}(\mathfrak{a})[\Sigma^{-1}]$$

where Σ is the class of (pointwise) quasi-isomorphisms [20]. In order to work conveniently with $D(\mathfrak{a})$ it is useful to introduce model structures [18] on $\mathsf{Dif}(\mathfrak{a})$.

It turns out that $\mathsf{Dif}(\mathfrak{a})$ is equipped with two canonical model structures for which the weak equivalences are the quasi-isomorphisms. [17]. For the *projective* model structure the fibrations are the pointwise epimorphisms and for the *injective* model structure the cofibrations are the pointwise monomorphisms.

We say $M \in \mathsf{Dif}(\mathfrak{a})$ is *fibrant* if $M \longrightarrow 0$ if a fibration for the injective model structure and we call M *cofibrant* if $0 \longrightarrow M$ is a cofibration for the projective model structure.

As usual $D(\mathfrak{a})$ is the homotopy category of cofibrant complexes and also the homotopy category of fibrant complexes and this makes it easy to construct left and right derived functors.

Let $\mathfrak{b}$ be another small DG-category and let $f : \mathfrak{a} \longrightarrow \mathfrak{b}$ be a DG-functor. f is a *quasi-equivalence* if $H^*(f)$ is fully faithful and $H^0(f)$ is essentially surjective and it is called a *quasi-isomorphism* if $H^*(f)$ is an isomorphism.

The functor f induces the usual triple of adjoint functors $(f^*, f_*, f^!)$ between $\mathsf{Dif}(\mathfrak{a})$ and $\mathsf{Dif}(\mathfrak{b})$. We denote the corresponding adjoint functors between $D(\mathfrak{a})$ and $D(\mathfrak{b})$ by $(Lf^*, f_*, Rf^!)$.

Proposition 2.2.1. [20] *The functors $(Lf^*, f_*, Rf^!)$ are equivalences when f is a quasi-equivalence.*

For $M, N \in \mathsf{Dif}(\mathfrak{a})$, $\mathrm{Hom}_{\mathfrak{a}}(M, N)$ denotes $\mathsf{Dif}(\mathfrak{a})(M, N) \in C(k)$. There is a corresponding derived functor

$$\mathrm{RHom}_{\mathfrak{a}} : D(\mathfrak{a})^{\mathrm{op}} \times D(\mathfrak{a}) \longrightarrow D(k).$$

Lemma 2.2.2. *If f is a quasi-equivalence then the induced map*

$$f_* : \mathrm{RHom}_{\mathfrak{a}}(M, N) \longrightarrow \mathrm{RHom}_{\mathfrak{b}}(f_*M, f_*N)$$

is a quasi-isomorphism.

Proof. After replacing M by a cofibrant resolution, f_* is defined as the composition

$$\mathrm{RHom}_{\mathfrak{a}}(M, N) = \mathrm{Hom}_{\mathfrak{a}}(M, N) \longrightarrow \mathrm{Hom}_{\mathfrak{b}}(f_*M, f_*N) \xrightarrow{\mathrm{can}} \mathrm{RHom}_{\mathfrak{b}}(f_*M, f_*N)$$

and looking at homology we see, using Proposition 2.2.1, that this is a quasi-isomorphism. $\qquad\qquad\square$

2.3. **Bimodules and resolutions.** An $\mathfrak{a}-\mathfrak{b}$-(DG-)bimodule X is an object of $\mathsf{Dif}(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b})$ which will be denoted by $(B, A) \longmapsto X(B, A)$, where $X(B, A)$ is contravariant in B and covariant in A. The $\mathfrak{a}-\mathfrak{a}$-bimodule $(A, A') \longmapsto \mathfrak{a}(A, A')$ will be denoted by $\mathfrak{a}$. For $\mathfrak{a}-\mathfrak{b}$-bimodules X and Y, we define the $\mathfrak{a}-\mathfrak{a}$ and $\mathfrak{b}-\mathfrak{b}$-bimodules

$$\begin{aligned}
(2.1) \qquad & \mathrm{Hom}_{\mathfrak{b}}(X, Y)(A, A') = \mathrm{Hom}_{\mathfrak{b}}(X(-, A), Y(-, A')) \\
& \mathrm{Hom}_{\mathfrak{a}^{\mathrm{op}}}(X, Y)(B, B') = \mathrm{Hom}_{\mathfrak{a}^{\mathrm{op}}}(X(B', -), Y(B, -)).
\end{aligned}$$

For $G \in \mathsf{Dif}(\mathfrak{b})$ and $F \in \mathsf{Dif}(\mathfrak{b}^{\mathrm{op}})$, there is also a tensor product $G \otimes_{\mathfrak{b}} F \in C(k)$ with in particular $\mathfrak{b}(-, B) \otimes_{\mathfrak{b}} F = F(B)$ and $G \otimes_{\mathfrak{b}} \mathfrak{b}(B, -) = G(B)$ (see for example [10, 14.3, 14.4]). For $X \in \mathsf{Dif}(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b})$ and $Y \in \mathsf{Dif}(\mathfrak{b}^{\mathrm{op}} \otimes \mathfrak{c})$, we define $X \otimes_{\mathfrak{b}} Y \in \mathsf{Dif}(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{c})$ by

$$(X \otimes_{\mathfrak{b}} Y)(C, A) = X(-, A) \otimes_{\mathfrak{b}} Y(C, -)$$

(see also [10, 14.5]).

If k is not a field then in the derived setting the tensor product $\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b}$ is philosophically wrong! Instead one should use something like "$\mathfrak{a}^{\mathrm{op}} \overset{L}{\otimes} \mathfrak{b}$" but of course this has no immediate meaning. Thus when working with bimodules we should assume that our categories satisfy an appropriate flatness assumption. It turns out that it is most convenient to assume that our categories are k-cofibrant in the following sense [19]:

Definition 2.3.1. A DG-category is *k-cofibrant* if all its Hom-sets are cofibrant in $C(k)$.

Recall that if $M \in C(k)$ is cofibrant then it has projective terms and the functors $\mathrm{Hom}(M, -)$ and $M \otimes -$ preserve acyclic complexes. This is technically very useful.

That Definition 2.3.1 is a good definition follows from the following result.

Proposition-Definition 2.3.2.   (1) *Let $\mathfrak{a}$ be a small DG-category. There exists a quasi-isomorphism $\bar{\mathfrak{a}} \longrightarrow \mathfrak{a}$ with $\bar{\mathfrak{a}}$ k-cofibrant which is surjective on Hom-sets (in the graded category). We call such $\bar{\mathfrak{a}} \longrightarrow \mathfrak{a}$ a k-cofibrant resolution of $\mathfrak{a}$.*

  (2) *If $f : \mathfrak{a} \longrightarrow \mathfrak{b}$ is a DG-functor between small DG-categories and $\bar{\mathfrak{b}} \longrightarrow \mathfrak{b}$ is a k-cofibrant resolution of $\mathfrak{b}$ then there exists a k-cofibrant resolution $\bar{\mathfrak{a}} \longrightarrow \mathfrak{a}$ together with a commutative diagram*

$$
\begin{array}{ccc}
\bar{\mathfrak{a}} & \xrightarrow{\ \bar{f}\ } & \bar{\mathfrak{b}} \\
\downarrow & & \downarrow \\
\mathfrak{a} & \xrightarrow[\ f\ ]{} & \mathfrak{b}
\end{array}
$$

Proof.   (1) We may take $\bar{\mathfrak{a}}$ to be a "semi-free" resolution of $\mathfrak{a}$. See [10, Lemma 13.5] and [17].

  (2) Again we let $\bar{\mathfrak{a}} \longrightarrow \mathfrak{a}$ be a semi-free resolution. The result now follows from [10, Lemma 13.6]. $\qquad\square$

If $\mathfrak{a}$, $\mathfrak{b}$ are small DG-categories then the derived category of $\mathfrak{a}$-$\mathfrak{b}$-modules should be defined as

$$(2.2) \qquad\qquad D(\bar{\mathfrak{a}}^{\mathrm{op}} \otimes \bar{\mathfrak{b}})$$

for k-cofibrant resolutions $\bar{\mathfrak{a}} \longrightarrow \mathfrak{a}$, $\bar{\mathfrak{b}} \longrightarrow \mathfrak{b}$. Propositions 2.2.1 and 2.3.2 insure that this definition is independent of the chosen resolutions (up to equivalence). We use this definition in principle, but to make things not overly abstract we will always indicate the resolutions $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{b}}$ explicitly in the notations.

Proposition 2.3.3. *Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are k-cofibrant. The derived functors*

$$\mathrm{RHom}_{\mathfrak{b}} : D(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b})^{\mathrm{op}} \times D(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b}) \longrightarrow D(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{a})$$

$$\mathrm{RHom}_{\mathfrak{a}^{\mathrm{op}}} : D(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b})^{\mathrm{op}} \times D(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b}) \longrightarrow D(\mathfrak{b}^{\mathrm{op}} \otimes \mathfrak{b}).$$

of (2.1) *may be computed pointwise in the sense*

$$(2.3) \qquad \begin{aligned} \mathrm{RHom}_{\mathfrak{b}}(X, Y)(A, A') &= \mathrm{RHom}_{\mathfrak{b}}(X(-, A), Y(-, A')) \\ \mathrm{RHom}_{\mathfrak{a}^{\circ}}(X, Y)(B, B') &= \mathrm{RHom}_{\mathfrak{a}^{\circ}}(X(B', -), Y(B, -)). \end{aligned}$$

Proof. Easy. $\qquad\square$

2.4. **Hochschild cohomology of DG-categories.** Let $\mathfrak{a}$ be a small k-cofibrant DG-category and let M be a $\mathfrak{a}$-bimodule. The Hochschild complex $\mathbf{C}(\mathfrak{a}, M)$ of $\mathfrak{a}$ with coefficients in an $\mathfrak{a}-\mathfrak{a}$-bimodule M is the total complex of the double complex $\mathbf{D}(\mathfrak{a}, M)$ with p-th column given by

$$(2.4) \qquad \prod_{A_0, \ldots, A_p} \mathrm{Hom}_k(\mathfrak{a}(A_{p-1}, A_p) \otimes_k \cdots \otimes_k \mathfrak{a}(A_0, A_1), M(A_0, A_p))$$

and the usual horizontal Hochschild differential. The Hochschild complex of $\mathfrak{a}$ is $\mathbf{C}(\mathfrak{a}) = \mathbf{C}(\mathfrak{a}, \mathfrak{a})$. There is an isomorphism in $D(k)$

$$(2.5) \qquad\qquad \mathbf{C}(\mathfrak{a}, M) \cong \mathrm{RHom}_{\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{a}}(\mathfrak{a}, M).$$

The Hochschild complex satisfies a "limited functoriality" property. If $j : \mathfrak{a} \longrightarrow \mathfrak{b}$ is a fully faithful map between small k-cofibrant DG-categories then there is an associated map between Hochschild complexes

$$j^* : \mathbf{C}(\mathfrak{b}) \longrightarrow \mathbf{C}(\mathfrak{a})$$

given by restricting cocycles. We will usually refer to j^* as the *restriction map*.

It is well-known that $\mathbf{C}(\mathfrak{a})$ carries a considerable amount of "higher structure" containing in particular the classical cup-product and the Gerstenhaber bracket. This extra structure is important for deformation theory. A convenient way of summarizing the extra structure is by saying that $\mathbf{C}(\mathfrak{a})$ is an algebra over the B_∞-operad [16, 19] which is an enlargement of the A_∞-operad. The map j^* introduced above is trivially compatible with the B_∞-structure.

As the Hochschild complex involves bimodules, according to the principles outlined in §2.3, its definition should be modified for non k-cofibrant DG-categories. The appropriate modification was introduced by Shukla and Quillen [33, 37] in the case of DG-algebras.

Let $\mathfrak{a}$ be a small DG-category which is not necessarily k-cofibrant and let M be in $\mathsf{Dif}(\mathfrak{a}^\circ \otimes \mathfrak{a})$. We fix a k-cofibrant resolution $\bar{\mathfrak{a}} \longrightarrow \mathfrak{a}$ and we define the "Shukla"-complex [37] of $\mathfrak{a}$ as

$$\mathbf{C}_{\mathrm{sh}}(\mathfrak{a}, M) = \mathbf{C}(\bar{\mathfrak{a}}, M)$$

and as usual $\mathbf{C}_{\mathrm{sh}}(\mathfrak{a}) = \mathbf{C}_{\mathrm{sh}}(\mathfrak{a}, \mathfrak{a})$.

Proposition 2.4.1. $\mathbf{C}_{\mathrm{sh}}(\mathfrak{a}, M)$ *is independent of* $\bar{\mathfrak{a}}$ *in* $D(k)$.

Proof. This follows from (2.5) (applied to $\bar{\mathfrak{a}}$) together with Lemma 2.2.2 and Proposition 2.3.2. The detailed proof is left to the reader. $\square$

Proposition 2.4.1 implies that $\mathbf{C}_{\mathrm{sh}}(\mathfrak{a})$ is well-defined in $D(k)$ which is rather weak. In §4.2 we will use results of Keller [19] to explain why $\mathbf{C}_{\mathrm{sh}}(\mathfrak{a})$ is well-defined in the homotopy category $\mathrm{Ho}(B_\infty)$ of B_∞-algebras and enjoys some functoriality properties extending the "limited functoriality" mentioned above.

2.5. **Non-small categories.** The definition of $\mathbf{C}_{\mathrm{sh}}(\mathfrak{a}, M)$ involves products of abelian groups which are indexed by tuples of objects of $\mathfrak{a}$. This creates a minor set theoretic problem if $\mathfrak{a}$ is not $\mathcal{U}$-small. Therefore in such a situation we will (implicitly) select a larger universe $\mathcal{V} \supset \mathcal{U}$ such that $\mathfrak{a}$ is $\mathcal{V}$-small. It is clear from (2.4) that the resulting Hochschild complex is, up to isomorphism, independent of the universe $\mathcal{V}$.

Remark 2.5.1. In the situations we encounter below $\mathbf{C}_{\mathrm{sh}}(\mathfrak{a}, M)$ will always have $\mathcal{U}$-small cohomology even if $\mathfrak{a}$ is not $\mathcal{U}$-small (although this will not always be obvious). Hence $\mathbf{C}_{\mathrm{sh}}(\mathfrak{a}, M)$ will always be $\mathcal{U}$-small in a homotopy theoretic sense.

3. Hochschild cohomology and deformation theory of abelian categories

If $\mathfrak{u}$ is a small linear category then $\mathsf{Mod}(\mathfrak{u}) = \mathsf{Add}(\mathfrak{u}^{\mathrm{op}}, \mathsf{Mod}(k))$ is the category of right $\mathfrak{u}$-modules.[1]

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[1]In [30] $\mathsf{Mod}(\mathfrak{u})$ was used to denote the category of *left* $\mathfrak{u}$-modules. The category of right $\mathfrak{u}$-modules was denoted by $\mathsf{Pr}(\mathfrak{u})$.

Assume that $\mathcal{A}$ is a small k-linear abelian category and let $\mathcal{M}$ be an object in $\mathsf{Mod}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})$. The category $\mathcal{C} = \mathsf{Ind}\mathcal{A}$ is the formal closure of $\mathcal{A}$ under filtered small colimits. We extend $\mathcal{M}$ to an object $\tilde{\mathcal{M}}$ in $\mathsf{Mod}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C})$ by

$$\tilde{\mathcal{M}}(\underset{i \in I}{\mathrm{inj} \lim} A_i, \underset{j \in J}{\mathrm{inj} \lim} B_j) = \underset{i \in I}{\mathrm{proj} \lim} \, \underset{j \in J}{\mathrm{inj} \lim} \, M(A_i, B_j)$$

It is well-known that $\mathcal{C}$ is a Grothendieck category and in particular it has enough injectives. Let $\mathfrak{i} = \mathsf{Inj}(\mathcal{C})$ be the category of injectives of $\mathcal{C}$. We restrict $\tilde{\mathcal{M}}$ to an $\mathfrak{i}$-bimodule M and we define the Hochschild complex of $\mathcal{M}$ as

$$(3.1) \qquad\qquad \mathbf{C}_{\mathrm{ab}}(\mathcal{A}, \mathcal{M}) = \mathbf{C}_{\mathrm{sh}}(\mathfrak{i}, M)$$

and $\mathbf{C}_{\mathrm{ab}}(\mathcal{A}) = \mathbf{C}_{\mathrm{ab}}(\mathcal{A}, \mathcal{A})$ (note that $\mathfrak{i}$ is not $\mathcal{U}$-small!) Definition (3.1) is motivated by the deformation theory of abelian categories which was introduced in [30]. For the convenience of the reader we now sketch this theory in order to show its relation to (3.1).

Below we assume that k is coherent. The coherentness assumption in necessary in deformation theory for technical reasons (see [30]) but it it *not* necessary for the definition of the Hochschild complex (3.1).

To start we need a notion of flatness. We say that $\mathcal{A}$ is *flat* if the injectives in $\mathcal{C}$ are k-flat in $\mathcal{C}^{\circ}$. We refer to [30, §3] for several equivalent (but more technical) characterizations of flatness which are intrinsic in terms of $\mathcal{A}$ (in particular from one of those characterizations it follows that flatness is self dual). Below we assume that $\mathcal{A}$ is flat.

Let Rng^0 be the category whose objects are coherent rings and whose morphisms are surjective maps with finitely generated nilpotent kernel and let $\theta : l \longrightarrow k$ be an object in Rng^0/k. A *flat l-deformation* of $\mathcal{A}$ is an l-linear flat abelian category $\mathcal{B}$ together with an equivalence $\mathcal{B}_k \cong \mathcal{A}$. Here $\mathcal{B}_k$ is the full-subcategory of $\mathcal{B}$ whose objects are annihilated by $\ker \theta$.

Let $\mathrm{Def}_{\mathcal{A}}(l)$ be the groupoid whose objects are the flat l-deformations of $\mathcal{A}$ and whose morphisms are the equivalences of deformations (in an obvious sense) up to natural isomorphism. The following result shows that Hochschild cohomology describes the obstruction theory for the functor

$$\mathrm{Def}_{\mathcal{A}} : \mathsf{Rng}^0/k \longrightarrow \mathsf{Gd}$$

Theorem 3.1. *Let $\sigma : l' \longrightarrow l$ be a map in Rng^0/k such that $I = \ker \sigma$ is annihilated by $\ker(l' \longrightarrow k)$. Let $\mathcal{B} \in \mathrm{Def}_{\mathcal{A}}(l)$ and let $\mathrm{Def}_{\mathcal{A}}(\sigma)^{-1}(\mathcal{B})$ be the groupoid whose objects are flat l'-deformations of $\mathcal{B}$.*

(1) *There is an obstruction $o(\mathcal{B}) \in \mathrm{HC}^3_{\mathrm{ab}}(\mathcal{A}, I \otimes \mathcal{A})$ such that $\mathrm{Ob}(\mathrm{Def}_{\mathcal{A}}(\sigma)^{-1}(\mathcal{B})) \neq \varnothing \iff o(\mathcal{B}) = 0$*

(2) *If $o(\mathcal{B}) = 0$ then $\mathrm{Sk}(\mathrm{Def}_{\mathcal{A}}(\sigma)^{-1}(\mathcal{B}))$ is an affine space over $\mathrm{HC}^2_{\mathrm{ab}}(\mathcal{A}, I \otimes \mathcal{A})$*

We now sketch the proof of this theorem. The proof will be complete in the case where k is a field (and this is already sufficient motivation for (3.1)). In the general case the proof depends on some results concerning the deformation theory of DG-categories which are well-known to experts but which do not seem to have appeared in the literature yet. We refer to [12, 27].

Assume that $\mathfrak{u}$ is a k-linear category. We say that $\mathfrak{u}$ is flat if all Hom-sets of $\mathfrak{u}$ are k-flat. Assume that $\mathfrak{u}$ is flat. A flat l-deformation of $\mathfrak{u}$ is a flat l-linear category $\mathfrak{v}$ together with an equivalence $\mathfrak{v} \otimes_l k \cong \mathfrak{u}$ where $\mathfrak{v} \otimes_l k$ is obtained by tensoring the Hom-sets of $\mathfrak{v}$ with k. As above the morphisms between deformations are

equivalences, up to natural isomorphism. The corresponding groupoid is denoted by $\operatorname{def}_{\mathfrak{u}}(l)$. The groupoid $\operatorname{def}_{\mathfrak{u}}^s(l)$ of *strict* deformations of $\mathfrak{u}$ is defined similarly except that we replace "equivalence" by "isomorphism" everywhere.

We recall the following:

Proposition 3.2. [30, Thm B.4] *The natural functor $\operatorname{def}_{\mathfrak{u}}^s(l) \longrightarrow \operatorname{def}_{\mathfrak{u}}(l)$ defines a bijection between the corresponding skeletons.*

As above let $\mathfrak{i}$ be the category of injectives in $\mathcal{C}$. The following is one of the main results of [30].

Theorem 3.3. [30, Thm. 8.8,Thm. 8.17] *The category $\mathfrak{i}$ is flat as linear category and there is an equivalence of categories between $\operatorname{Def}_{\mathcal{A}}(l)$ and $\operatorname{def}_{\mathfrak{i}}(l)$.*

It follows that the deformation theory of abelian categories reduces to the deformation theory of linear categories.

Let $\mathfrak{u}$ be a flat k-linear category as above. If k is a field then it is well-known that (as in the algebra case) the strict deformation theory of $\mathfrak{u}$ is controlled by the Hochschild cohomology of $\mathfrak{u}$. Let $\sigma : l' \longrightarrow l$ be a map in Rng^0/k such that $I = \ker \sigma$ is annihilated by $\ker(l' \longrightarrow k)$ and let $\mathfrak{v}$ be a flat l-deformation of $\mathfrak{u}$. Then it follows from Proposition 3.2 (replacing k by l and l by l') that there is a bijection $\operatorname{Sk}(\operatorname{def}_{\mathfrak{u}}^s(\sigma)^{-1}(\mathfrak{v})) \cong \operatorname{Sk}(\operatorname{def}_{\mathfrak{u}}(\sigma)^{-1}(\mathfrak{v}))$. Hence the non-strict deformation theory of $\mathfrak{u}$ is controlled by Hochschild cohomology as well and this then leads to a proof of (3.1).

If k is not a field then there is a technical difficulty in the sense that when the Hom-sets of $\mathfrak{u}$ are not projective over k, the naive Hochschild cohomology of $\mathfrak{u}$, $\mathbf{HC}^*(\mathfrak{u}, I \otimes \mathfrak{u})$ does *not* lead to the correct obstruction theory for the deformations of $\mathfrak{u}$. This problem is rather serious since the linear category $\mathfrak{i} = \mathsf{Inj}(\mathcal{C})$ will have flat, but not in general projective Hom-sets.

Nevertheless it is true that the deformation theory of $\mathfrak{u}$ is controled by $\mathbf{HC}_{\mathrm{sh}}^*(\mathfrak{u})$ [27]. To prove this one replaces $\mathfrak{u}$ by an appropriate semi-free resolution [10, Lemma 13.5] $\bar{\mathfrak{u}}$ and one studies the deformations of $\bar{\mathfrak{u}}$ in the homotopy category of DG-categories which turns out to be controlled by the homology of $\operatorname{Der}(\bar{\mathfrak{u}}, I \otimes \bar{\mathfrak{u}})$ in degree 1 and 2. There is an exact triangle

$$\operatorname{Der}(\bar{\mathfrak{u}}, I \otimes \bar{\mathfrak{u}})[-1] \longrightarrow \mathbf{C}(\bar{\mathfrak{u}}, I \otimes \bar{\mathfrak{u}}) \longrightarrow I \otimes \bar{\mathfrak{u}} \longrightarrow$$

and therefore in degrees ≥ 2, $\mathbf{C}(\bar{\mathfrak{u}}, I \otimes \bar{\mathfrak{u}})$ and $\operatorname{Der}(\bar{\mathfrak{u}}, I \otimes \bar{\mathfrak{u}})[-1]$ have the same homology which finished the proof.

Remark 3.4. If $\mathfrak{u}$ is flat and M is a $\mathfrak{u}$ bimodule then it is not hard to see that there is a quasi-isomorphism in $D(k)$

$$\mathbf{C}_{\mathrm{sh}}(\mathfrak{u}, M) = \operatorname{RHom}_{\mathfrak{u}^{\mathrm{op}} \otimes \mathfrak{u}}(\mathfrak{u}, M)$$

This "more elementary" interpretation of Shukla cohomology does not seem to be useful for deformation theory however.

Remark 3.5. If k is a field of characteristic zero then it follows from Proposition 3.2 and Theorem 3.3 that there is a bijection between $\operatorname{Sk}(\operatorname{Def}_{\mathcal{A}}(l))$ and the solutions of the Maurer-Cartan equation in $\mathbf{C}_{\mathrm{ab}}(\mathcal{A})[1]$ with coefficients in $\ker(l \longrightarrow k)$, modulo gauge equivalence [23]. Hence $\mathbf{C}_{\mathrm{ab}}(\mathcal{A})[1]$ is the DG-Lie algebra controlling the deformation theory of $\mathcal{A}$.

Remark 3.6. Assume that k is a field. If $\dim \mathbf{HC}^2_{\mathrm{ab}}(\mathcal{A}) < \infty$ then $\mathcal{A}$ has a (formal) versal deformation [2, 38, Chapt. 6]. Or in equivalent terms: the functor

$$\mathrm{Sk}(\mathrm{Def}_{\mathcal{A}}(-)) : \mathsf{Rng}^0/k \longrightarrow \mathsf{Set}$$

has a hull [36]. I.e. there is a noetherian local ring (R, m) with residue field k together with compatible flat deformations $\mathcal{A}_n$ of $\mathcal{A}$ over R/m^n such that the formal object $\mathrm{proj}\lim_n \mathcal{A}_n$ in $\mathrm{Def}_{\mathcal{A}}(R)$ satisfies the versality condition. Using the existence criterion in [36] this can be shown by translation to the linear case using Proposition 3.2 and Theorem 3.3.

Of course one actually wants an R-linear abelian category $\tilde{\mathcal{A}}$ representing the formal object $\mathrm{proj}\lim_n \mathcal{A}_n$. This is possible under certain conditions. See [28].

Remark 3.7. In the rest of this paper we restrict ourselves to the study of $\mathbf{C}_{\mathrm{ab}}(\mathcal{A})$ as this is the most interesting case. And in any case this is sufficient for deformation theory in case k is a field.

4. More background on Hochschild cohomology of DG-categories

As we have seen the Hochschild cohomology of an abelian category is basically the Hochschild cohomology of a suitable DG-category. So we need techniques for computing the Hochschild cohomology of DG-categories. A powerful tool in this respect was provided by Keller in [19].

4.1. **Keller's results.** All results in this section are due to Keller. See [19]. Suppose we have small k-cofibrant DG-categories $\mathfrak{a}$ and $\mathfrak{b}$ and suppose that X is a cofibrant $\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b}$-module. Then by functoriality, we get a map of $\mathfrak{a}-\mathfrak{a}$-bimodules

$$\lambda : \mathfrak{a} \longrightarrow \mathrm{Hom}_{\mathfrak{b}}(X, X) = \mathrm{RHom}_{\mathfrak{b}}(X, X)$$

and an induced map

$$\lambda_* : \mathbf{C}(\mathfrak{a}) \longrightarrow \mathbf{C}(\mathfrak{a}, \mathrm{Hom}_{\mathfrak{b}}(X, X))$$

Similarly,

$$\omega : \mathfrak{b} \longrightarrow \mathrm{Hom}_{\mathfrak{a}^{\mathrm{op}}}(X, X) = \mathrm{RHom}_{\mathfrak{a}^{\mathrm{op}}}(X, X)$$

induces

$$\omega_* : \mathbf{C}(\mathfrak{b}) \longrightarrow \mathbf{C}(\mathfrak{b}, \mathrm{Hom}_{\mathfrak{a}^{\mathrm{op}}}(X, X))$$

Below, in case X is variable, we will adorn λ and ω by a subscript X.

Let $\mathfrak{c}$ be the DG-category such that

$$\mathrm{Ob}(\mathfrak{c}) = \mathrm{Ob}(\mathfrak{a}) \coprod \mathrm{Ob}(\mathfrak{b})$$

and such that

$$\mathfrak{c}(U, V) = \begin{cases} \mathfrak{a}(U, V) & \text{if } U, V \in \mathfrak{a} \\ \mathfrak{b}(U, V) & \text{if } U, V \in \mathfrak{b} \\ X(U, V) & \text{if } U \in \mathfrak{b}, V \in \mathfrak{a} \\ 0 & \text{otherwise} \end{cases}$$

Below we will sometimes use the notation $(\mathfrak{a} \xleftarrow{X} \mathfrak{b})$ for the category $\mathfrak{c}$ and we refer to $\mathfrak{c}$ as an "arrow category".

Consider the canonical inclusions $i_{\mathfrak{a}} : \mathfrak{a} \longrightarrow \mathfrak{c}$ and $i_{\mathfrak{b}} : \mathfrak{b} \longrightarrow \mathfrak{c}$.

Theorem 4.1.1. [19, §4.4,4.5]
  (1) *There is a quasi-isomorphism* $\mathbf{C}(\mathfrak{a}, \mathrm{RHom}_{\mathfrak{b}}(X, X)) \cong \mathbf{C}(\mathfrak{b}, \mathrm{RHom}_{\mathfrak{a}^{\mathrm{op}}}(X, X))$.

(2) *If λ_* is a quasi-isomorphism, $i_{\mathfrak{b}}^* : \mathbf{C}(\mathfrak{c}) \longrightarrow \mathbf{C}(\mathfrak{b})$ is a quasi-isomorphism of B_∞-algebra's.*

(3) *If ω_* is a quasi-isomorphism, $i_{\mathfrak{a}}^* : \mathbf{C}(\mathfrak{c}) \longrightarrow \mathbf{C}(\mathfrak{a})$ is a quasi-isomorphism of B_∞-algebra's.*

So in particular if λ_* is a quasi-isomorphism then we have an induced map in $\mathrm{Ho}(B_\infty)$

$$\phi_X : \mathbf{C}(\mathfrak{b}) \longrightarrow \mathbf{C}(\mathfrak{a})$$

given by $i_{\mathfrak{a}}^*(i_{\mathfrak{b}}^*)^{-1}$. If ω_* is a quasi-isomorphism then ϕ_X is an isomorphism.

Theorem 4.1.2. [19]

(1) *ϕ_X depends only on the isomorphism class of X in $D(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b})$.*
(2) *If $j : \mathfrak{a} \longrightarrow \mathfrak{b}$ is fully faithful and X is given by $X(B, A) = \mathfrak{a}(B, j(A))$ then λ is a quasi-isomorphism and $\phi_X = j^*$ in $\mathrm{Ho}(B_\infty)$.*

Proof. 1. is proved as [19, Thm. 4.6a]. 2. is [19, Thm. 4.6c]. $\qquad\square$

Theorem 4.1.3. [19, Thm 4.6b] *If $- \otimes_{\mathfrak{a}} X$ induces a fully faithful functor $D(\mathfrak{a}) \longrightarrow D(\mathfrak{b})$ then λ is a quasi-isomorphism and hence there is a well defined associated "restriction" morphism $\phi_X : \mathbf{C}(\mathfrak{b}) \longrightarrow \mathbf{C}(\mathfrak{a})$. If $- \otimes_{\mathfrak{a}} X$ induces an equivalence then ϕ_X is an isomorphism.*

There is also a transitivity result for the maps ϕ_X.

Theorem 4.1.4. [19] *Let X be a cofibrant $\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b}$-module such that $(\lambda_X)_*$ is a quasi-isomorphism and let Y be a cofibrant $\mathfrak{b}^{\mathrm{op}} \otimes \mathfrak{c}$-module inducing a fully faithful functor $D(\mathfrak{b}) \longrightarrow D(\mathfrak{c})$. Then $(\lambda_{X \otimes_{\mathfrak{b}} Y})_*$ is an isomorphism and $\phi_{X \otimes_{\mathfrak{b}} Y} = \phi_Y \circ \phi_X$.*

Proof. This is proved as [19, Thm 4.6d]. $\qquad\square$

Remark 4.1.5. If $j : \mathfrak{a} \longrightarrow \mathfrak{b}$ is fully faithful then instead of the $\mathfrak{a} - \mathfrak{b}$-module X defined in Theorem 4.1.2 it is just as natural to use the $\mathfrak{b} - \mathfrak{a}$ bimodule Y defined by $Y(A, B) = \mathrm{Hom}_{\mathfrak{b}}(j(A), B)$. One may dualize all arguments for this bimodule. The relevant arrow category is now $\mathfrak{c}' = (\mathfrak{b} \xleftarrow{Y} \mathfrak{a})$ where the arrow is just the inclusion. It turns our that now ω_Y is a quasi-isomorphism. Thus we obtain a morphism in $\mathrm{Ho}(B_\infty)$ given by $i_{\mathfrak{b}}^*(i_{\mathfrak{a}}^*)^{-1}$. Dualizing the proof of Theorem 4.1.2.2 above we still find $i_{\mathfrak{b}}^*(i_{\mathfrak{a}}^*)^{-1} = j^*$ in $\mathrm{Ho}(B_\infty)$. Hence the "bimodule interpretation" of limited functoriality is unambiguous.

4.2. **The functoriality of the Shukla complex.** Now we do not assume that our DG-categories are k-cofibrant. Using the results in §4.1 we can explain why the Shukla complex is well-defined and functorial. More precisely we show that $\mathbf{C}_{\mathrm{sh}}(-) : \mathfrak{a} \longrightarrow \mathbf{C}_{\mathrm{sh}}(\mathfrak{a})$ defines a contravariant functor on a suitable category of small DG-categories with values in $\mathrm{Ho}(B_\infty)$.

Let $\mathcal{F}$ be the category whose objects are small DG-categories. If $\mathfrak{a}, \mathfrak{b} \in \mathrm{Ob}(\mathcal{F})$ then $\mathcal{F}(\mathfrak{a}, \mathfrak{b})$ is defined as the set of equivalence classes of triples $(\bar{\mathfrak{a}}, X, \bar{\mathfrak{b}})$ where $\bar{\mathfrak{a}} \longrightarrow \mathfrak{a}$, $\bar{\mathfrak{b}} \longrightarrow \mathfrak{b}$ are k-cofibrant resolutions and X is a cofibrant $\bar{\mathfrak{a}}^{\mathrm{op}} \otimes \bar{\mathfrak{b}}$-module such that $- \otimes_{\bar{\mathfrak{a}}} X$ induces a fully faithful functor $D(\mathfrak{a}) \cong D(\bar{\mathfrak{a}}) \longrightarrow D(\bar{\mathfrak{b}}) \cong D(\mathfrak{b})$. Two triples $(\bar{\mathfrak{a}}, X, \bar{\mathfrak{b}})$, $(\bar{\mathfrak{a}}', X', \bar{\mathfrak{b}}')$ are equivalent if X and X' correspond under the canonical equivalence between $D(\bar{\mathfrak{a}}^{\mathrm{op}} \otimes \bar{\mathfrak{b}})$ and $D(\bar{\mathfrak{a}}'^{\mathrm{op}} \otimes \bar{\mathfrak{b}}')$.

If $\bar{\mathfrak{a}} \longrightarrow \mathfrak{a}$, $\bar{\mathfrak{a}}' \longrightarrow \mathfrak{a}$ are k-cofibrant resolutions then we define $C_{\bar{\mathfrak{a}}\bar{\mathfrak{a}}'}$ to be a cofibrant $\bar{\mathfrak{a}}^{\mathrm{op}} \otimes_k \bar{\mathfrak{a}}'$ resolution of $\mathfrak{a}$ considered as $\bar{\mathfrak{a}}^{\mathrm{op}} - \bar{\mathfrak{a}}'$-bimodule. $(\bar{\mathfrak{a}}, C_{\bar{\mathfrak{a}}\bar{\mathfrak{a}}'}, \bar{\mathfrak{a}}')$

defines a canonical equivalence class of objects in $\mathcal{F}(\mathfrak{a}, \mathfrak{a})$ which we denote by $\mathrm{id}_{\mathfrak{a}}$ since $C_{\bar{\mathfrak{a}}\bar{\mathfrak{a}}'}$ induces the identity on $D(\mathfrak{a})$.

Composition of triples is defined as

$$(\bar{\mathfrak{a}}, X, \bar{\mathfrak{b}}) \circ (\bar{\mathfrak{b}}', Y, \bar{\mathfrak{c}}) = (\bar{\mathfrak{a}}, X \otimes_{\bar{\mathfrak{b}}} C'_{\bar{\mathfrak{b}}\bar{\mathfrak{b}}'} \otimes_{\bar{\mathfrak{b}}'} Y, \bar{\mathfrak{c}})$$

It is easy to see that this is compatible with equivalence. Furthermore the maps $\mathrm{id}_{\mathfrak{x}}$ for $\mathfrak{x} \in \mathrm{Ob}(\mathcal{F})$ behave as identities for this composition.

Now for every $\mathfrak{a} \in \mathrm{Ob}(\mathcal{F})$ fix a k-cofibrant resolution $\bar{\mathfrak{a}}$ and define $\mathbf{C}_{\mathrm{sh}}(\mathfrak{a}) = \mathbf{C}(\bar{\mathfrak{a}})$. We will make $\mathbf{C}_{\mathrm{sh}}(-)$ into a functor on $\mathcal{F}$. Assume we have a triple $(\bar{\mathfrak{a}}', X', \bar{\mathfrak{b}}')$. Then we have an equivalent triple of the form $(\bar{\mathfrak{a}}, X, \bar{\mathfrak{b}})$, with $X = C_{\bar{\mathfrak{a}}\bar{\mathfrak{a}}'} \otimes_{\bar{\mathfrak{a}}'} X' \otimes_{\bar{\mathfrak{b}}'} C_{\bar{\mathfrak{b}}'\bar{\mathfrak{b}}}$. To the triple $(\bar{\mathfrak{a}}', X', \bar{\mathfrak{b}}')$ we now associate the map $\phi_X : \mathbf{C}(\bar{\mathfrak{a}}) \longrightarrow \mathbf{C}(\bar{\mathfrak{b}})$. Using Theorems 4.1.2, 4.1.4 it is easy to see that the assignment $(\bar{\mathfrak{a}}', X', \bar{\mathfrak{b}}') \mapsto \phi_X$ is compatible with equivalence, compositions and sends $\mathrm{id}_{\mathfrak{a}}$ to the identity. In this way we have reached our goal of making $\mathbf{C}_{\mathrm{sh}}(-)$ into a functor.

Remark 4.2.1. The functor $\mathbf{C}_{\mathrm{sh}}(-)$ inverts quasi-equivalences. This follows from the corresponding result in the k-cofibrant case (see Theorem 4.1.3).

Remark 4.2.2. If we choose another system of resolutions $q : \bar{\mathfrak{a}}' \longrightarrow \mathfrak{a}$ then we have a canonical isomorphism $\mathbf{C}(\bar{\mathfrak{a}}) \longrightarrow \mathbf{C}(\bar{\mathfrak{a}}')$ induced by $C_{\bar{\mathfrak{a}}\bar{\mathfrak{a}}'}$ which defines a natural isomorphism between the functors $\mathfrak{a} \longmapsto \mathbf{C}(\bar{\mathfrak{a}})$ and $\mathfrak{a} \longmapsto \mathbf{C}(\bar{\mathfrak{a}}')$. So the functor $\mathbf{C}_{\mathrm{sh}}(-)$ is well defined up to a canonical natural isomorphism.

Remark 4.2.3. If $j : \mathfrak{a} \longrightarrow \mathfrak{b}$ is a fully faithful functor and if $\bar{\mathfrak{b}} \longrightarrow \mathfrak{b}$ is a k-cofibrant resolution of $\mathfrak{b}$ then we may restrict this resolution to a k-cofibrant resolution of $\mathfrak{a}$. So j extends to a fully faithful functor $\bar{j} : \bar{\mathfrak{a}} \longrightarrow \bar{\mathfrak{b}}$. Thus $\bar{j}$ defines a morphism of Shukla complexes

$$\mathbf{C}_{\mathrm{sh}}(\mathfrak{b}) \longrightarrow \mathbf{C}_{\mathrm{sh}}(\mathfrak{a})$$

which we will denote by j^*. This notation is natural by Remark 4.1.5.

4.3. **The "Cosmic Censorship" principle.** Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are small k-cofibrant DG-categories. It follows from Theorem 4.3.3 that $\mathbf{C}(\mathfrak{a}) \cong \mathbf{C}(\mathfrak{b})$ in $\mathrm{Ho}(B_{\infty})$ if ω and λ are quasi-isomorphims (and indeed this is the way the result is explicitly stated in [19]). λ and ω being quasi-isomorphisms is equivalent to

$$\lambda(A, A') : \mathfrak{a}(A, A') \longrightarrow \mathrm{RHom}_{\mathfrak{b}}(X(-, A), X(-, A'))$$
$$\omega(B, B') : \mathfrak{b}(B, B') \longrightarrow \mathrm{RHom}_{\mathfrak{a}^{\mathrm{op}}}(X(B', -), X(B, -))$$

being quasi-isomorphisms for all $A, A' \in \mathrm{Ob}(\mathfrak{a})$ and $B, B' \in \mathrm{Ob}(\mathfrak{b})$.

The extra generality of Theorem 4.1.1 will be essential for us when we study ringed spaces, for it turns out that sometimes λ_* or ω_* are quasi-isomorphisms when this is not necessarily the case for λ or ω.

In the application to ringed spaces $\mathrm{Ob}(\mathfrak{a})$ will be equipped with a non-trivial transitive relation $\mathcal{R}$ such that $\mathfrak{a}(A, A') = 0$ if $(A, A') \notin \mathcal{R}$. We will call such $\mathcal{R}$ a *censoring relation*. Note that any $\mathfrak{a}$ has a *trivial censoring relation* given by $\mathcal{R} = \mathrm{Ob}(\mathfrak{a}) \times \mathrm{Ob}(\mathfrak{a})$. We have the following result.

Proposition 4.3.1. *Assume that $\mathfrak{a}$ has a censoring relation $\mathcal{R}$ and let M be an $\mathfrak{a}-\mathfrak{a}$-bimodule. Define M_0 by*

$$M_0(A, A') = \begin{cases} M(A, A') & \text{if } (A, A') \in \mathcal{R} \\ 0 & \text{otherwise} \end{cases}$$

Then M_0 is a subbimodule of M and furthermore

$$\text{(4.1)} \qquad \mathbf{C}(\mathfrak{a}, M) = \mathbf{C}(\mathfrak{a}, M_0)$$

Proof. That M_0 is a submodule is clear. The equality $\mathbf{C}(\mathfrak{a}, M) = \mathbf{C}(\mathfrak{a}, M_0)$ follows immediately from the definition of the Hochschild complex. $\qquad\square$

Proposition 4.3.2. *Assume that $\mathfrak{a}$ has a censoring relation $\mathcal{R}$ and*

$$\lambda(A, A') : \mathfrak{a}(A, A') \longrightarrow \mathrm{RHom}_{\mathfrak{b}}(X(-, A), X(-, A'))$$

is an isomorphism for all $(A, A') \in \mathcal{R}$. Then λ_* is a quasi-isomorphism.

So in a sense the zero Hom-sets in $\mathfrak{a}$ censor possible "bad parts" of $\mathrm{RHom}_{\mathfrak{b}}(X, X)$. See Remark 7.3.2 below for an application of this principle.

Proof. Our hypotheses imply that λ factors as

$$\mathfrak{a}(A, A') \longrightarrow \mathrm{Hom}_{\mathfrak{b}}(X, X)_0 \longrightarrow \mathrm{Hom}_{\mathfrak{b}}(X, X)$$

and the first map is quasi-isomorphism. Hence λ_* factors as a composition of a quasi-isomorphism and an isomorphism

$$\mathbf{C}(\mathfrak{a}(A, A')) \longrightarrow \mathbf{C}(\mathfrak{a}, \mathrm{Hom}_{\mathfrak{b}}(X, X)_0) \longrightarrow \mathbf{C}(\mathfrak{a}, \mathrm{Hom}_{\mathfrak{b}}(X, X))$$

finishing the proof. $\qquad\square$

Now assume that $\mathfrak{a}$ and $\mathfrak{b}$ be are arbitrary small (not necessarily k-cofibrant) DG-categories which are equipped with (possibly trivial) censoring relations $\mathcal{R}$ and $\mathcal{L}$ and let X be an object in $\mathsf{Dif}(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b})$. We will use the following criterion to compare the Shukla complexes of $\mathfrak{a}$ and $\mathfrak{b}$.

Proposition 4.3.3. *Assume that the compositions*

$$\text{(4.2)} \quad \begin{array}{llll} \Lambda_{A,A'} : \mathfrak{a}(A,A') & \longrightarrow & \mathrm{Hom}_{\mathfrak{b}}(X(-,A), X(-,A')) & \xrightarrow{\mathrm{can}} & \mathrm{RHom}_{\mathfrak{b}}(X(-,A), X(-,A')) \\ \Omega_{B,B'} : \mathfrak{b}(B,B') & \longrightarrow & \mathrm{Hom}_{\mathfrak{a}^{\mathrm{op}}}(X(B',-), X(B,-)) & \xrightarrow{\mathrm{can}} & \mathrm{RHom}_{\mathfrak{a}^{\mathrm{op}}}(X(B',-), X(B,-)) \end{array}$$

are quasi-isomorphisms for all $(A, A') \in \mathcal{R}$ and $(B, B') \in \mathcal{L}$. Then $\mathbf{C}_{\mathrm{sh}}(\mathfrak{a}) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{b})$ in $\mathrm{Ho}(B_\infty)$.

Proof. Let $\bar{\mathfrak{a}} \longrightarrow \mathfrak{a}$ and $\bar{\mathfrak{b}} \longrightarrow \mathfrak{b}$ be k-cofibrant resolutions. By replacing $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{b}}$ by $\bar{\mathfrak{a}}_0$ and $\bar{\mathfrak{b}}_0$ respectively we may *and we will* assume that $\mathcal{R}$ and $\mathcal{L}$ are censoring relations for $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{b}}$.

Now let $\bar{X} \longrightarrow X$ be a cofibrant resolution of X in $\mathsf{Dif}(\bar{\mathfrak{a}}^{\mathrm{op}} \otimes \bar{\mathfrak{b}})$. We may apply Theorem 4.1.1 and Proposition 4.3.2 to the triple $(\bar{\mathfrak{a}}, \overline{X}, \bar{\mathfrak{b}})$. Thus we need to check that the appropriate $\lambda(A, A')$ and $\omega(B, B')$ are quasi-isomorphisms.

For $A, A' \in \mathrm{Ob}(\bar{\mathfrak{a}})$ we have a commutative diagram in $D(k)$

$$\begin{array}{ccc} \bar{\mathfrak{a}}(A, A') & \xrightarrow{\lambda(A, A')} & \mathrm{Hom}_{\bar{\mathfrak{b}}}(\bar{X}(-, A), \bar{X}(-, A')) \\ \cong \Big\downarrow & & \Big\downarrow \cong \\ \mathfrak{a}(A, A') & \xrightarrow[\Lambda_{A,A'}]{} & \mathrm{RHom}_{\mathfrak{b}}(X(-, A), X(-, A')) \end{array}$$

in which the quasi-isomorphism to the right is justified by Lemma 2.2.2. Hence $\lambda(A, A')$ is a quasi-isomorphism if this is true for the corresponding map in (4.2). A similar observation holds for $\omega(B, B')$. This finishes the proof. $\qquad\square$

We sometimes use the following compact criterion to see if a fully faithful map induces an isomorphism on Shukla cohomology.

Proposition 4.3.4. *Let $j : \mathfrak{a} \longrightarrow \mathfrak{b}$ be fully faithful and assume that $\mathfrak{b}$ has a (possibly trivial) censoring relation $\mathcal{L}$ such that for all $(B, B') \in \mathcal{R}$ the canonical maps given by functoriality*

$$\mathfrak{b}(B, B') \longrightarrow \mathrm{RHom}_{\mathfrak{a}^{\mathrm{op}}}(\mathrm{Hom}_{\mathfrak{b}}(B', j(-)), \mathrm{Hom}_{\mathfrak{b}}(B, j(-)))$$

are isomorphisms then $j^* : \mathbf{C}_{\mathrm{sh}}(\mathfrak{a}) \longrightarrow \mathbf{C}_{\mathrm{sh}}(\mathfrak{b})$ is an isomorphism in $\mathrm{Ho}(B_\infty)$.

Proof. As in Remark 4.2.3 we may lift $j : \mathfrak{a} \longrightarrow \mathfrak{b}$ to a fully faithfully map $\bar{j} : \bar{\mathfrak{a}} \longrightarrow \bar{\mathfrak{b}}$ between k-cofibrant resolutions of $\mathfrak{a}$ and $\mathfrak{b}$. Let $\bar{X}$ be a $\bar{\mathfrak{a}} - \bar{\mathfrak{b}}$-bimodule which is a cofibrant resolution of the bimodule given by $X(B, A) = \mathrm{Hom}_{\mathfrak{b}}(A, j(B))$. Then by Theorem 4.1.2 $\phi_{\bar{X}} = \bar{j}^*$ and λ is a quasi-isomorphism. So we have to check that ω_* is a quasi-isomorphism. As in the proof of Proposition 4.3.3 we may do this by checking that the $\Omega_{B,B'}$ are quasi-isomorphisms for $(B, B') \in \mathcal{L}$. But this is precisely the assertion of the current proposition. $\qquad\square$

In other words for the previous proposition to apply the contravariant representable functors corresponding to objects in $\mathfrak{b}$ should have the same RHom's as their restrictions to $\mathfrak{a}$. Note that Proposition 4.3.4 has a dual version using contravariant representable functors which we will also use below.

Convention. *Below, unless otherwise specified, if we write $\mathbf{C}_1 \cong \mathbf{C}_2$ for B_∞-algebras $\mathbf{C}_1$ and $\mathbf{C}_2$ we mean that $\mathbf{C}_1$ and $\mathbf{C}_2$ are isomorphic in $\mathrm{Ho}(B_\infty)$.*

4.4. **DG-categories of cofibrant objects.** In this section we give an easy but rather spectacular application of Proposition 4.3.4. A weak version of it will be used afterwards to compare the Hochschild cohomology of an abelian category and its bounded derived category (viewed as a DG-category).

Let $\mathfrak{a}$ be a small DG-category and let

$$\mathfrak{a} \longrightarrow \mathsf{Dif}(\mathfrak{a}) : \mathfrak{a} \longrightarrow \mathfrak{a}(-, A)$$

be the Yoneda embedding. We have the following

Theorem 4.4.1. *Let $\mathfrak{b}$ be any DG-subcategory of $\mathsf{Dif}(\mathfrak{a})$ which consists of cofibrant objects and which contains $\mathfrak{a}$. Then the restrictiom map $\mathbf{C}_{\mathrm{sh}}(\mathfrak{b}) \longrightarrow \mathbf{C}_{\mathrm{sh}}(\mathfrak{a})$ is a quasi-isomorphism.*

Proof. Let $j : \mathfrak{a} \longrightarrow \mathfrak{b}$ be the inclusion functor. By the dual version of Proposition 4.3.4 we need to prove

$$(4.3) \qquad \mathrm{RHom}_{\mathfrak{a}}(\mathfrak{b}(j(-), B), \mathfrak{b}(j(-), B')) = \mathfrak{b}(B, B') = \mathrm{Hom}_{\mathfrak{a}}(B, B')$$

for $B, B' \in \mathfrak{b}$. Since $\mathfrak{b}(j(-), B) = \mathrm{Hom}_{\mathfrak{a}}(j(-), B) = B$ for $B \in \mathfrak{b}$ (recall $\mathsf{Dif}(\mathfrak{a}) = \mathsf{DGFun}(\mathfrak{a}^{\mathrm{op}}, C(k))$), (4.3) follows from the fact that B is cofibrant. $\qquad\square$

4.5. **Hochschild cohomology as a derived center.** It is well known that the Hochschild cohomology of a ring may be regarded as a kind of non-additive derived version of the center. In this section we show that this generalizes trivially to Shukla cohomology of DG-categories. That is, we will construct for a small DG-category $\mathfrak{a}$ a canonically defined map

$$\sigma_{\mathfrak{a}} : H\mathbf{C}_{\mathrm{sh}}^*(\mathfrak{a}) \longrightarrow Z(H^*(\mathfrak{a}))$$

where $Z(H^*(\mathfrak{a}))$ is the center of the graded category $H^*(\mathfrak{a})$.

Assume that $\mathfrak{u}$ is a small k-linear $\mathbb{Z}$-graded category. The *center* $Z(\mathfrak{u})$ of $\mathfrak{u}$ is by definition the ring of graded endomorphisms of the identity functor on $\mathfrak{u}$. More concretely the center of $\mathfrak{u}$ is a graded ring whose homogeneous elements consists of tuples of homogeneous elements $(\phi_U)_U \in \prod_{U \in \mathfrak{u}} \mathfrak{u}(U, U)$ such that for any homogeneous $f \in \mathfrak{u}(U, V)$ one has $f\phi_U = (-1)^{|f||\phi_U|}\phi_V f$. In particular $Z(\mathfrak{u})$ is (super) commutative.

Assume that $\mathfrak{a}$ is a k-cofibrant DG-category. Let

$$\Sigma_{\mathfrak{a}} : \mathbf{C}(\mathfrak{a}) \longrightarrow \prod_{A \in \mathfrak{a}} \mathfrak{a}(A, A)$$

be the map associated to the morphism of double complexes which sends $\mathbf{D}(\mathfrak{a})$ to its first column. An easy computation with the explicit formulas for the cup product [16] shows that $\sigma_{\mathfrak{a}} \stackrel{\text{def}}{=} H^*(\Sigma_{\mathfrak{a}})$ is a graded ring map which maps $H\mathbf{C}^*(\mathfrak{a})$ to the center of $H^*(\mathfrak{a})$

Now let $\mathfrak{a}$ be an arbitrary small DG-category and let $\bar{\mathfrak{a}} \longrightarrow \mathfrak{a}$ be a k-cofibrant resolution of $\mathfrak{a}$. We define $\bar{\sigma}_{\bar{\mathfrak{a}}} : H\mathbf{C}^*_{\text{sh}}(\mathfrak{a}) \longrightarrow Z(H^*(\mathfrak{a}))$ as the composition

$$\mathbf{C}_{\text{sh}}(\mathfrak{a}) = \mathbf{C}(\bar{\mathfrak{a}}) \xrightarrow{\sigma_{\bar{\mathfrak{a}}}} Z(H^*(\bar{\mathfrak{a}})) \cong Z(H^*(\mathfrak{a})))$$

Let $\bar{\mathfrak{a}}' \longrightarrow \mathfrak{a}$ be another k-cofibrant resolution of $\mathfrak{a}$ and let $C_{\bar{\mathfrak{a}}\bar{\mathfrak{a}}'}$ be as in §4.2. Let $\bar{\mathfrak{c}} = (\bar{\mathfrak{a}} \xleftarrow{C_{\bar{\mathfrak{a}}\bar{\mathfrak{a}}'}} \bar{\mathfrak{a}}')$. Then

$$H^*(\bar{\mathfrak{c}}) = (H^*(\bar{\mathfrak{a}}) \xleftarrow{H^*(C_{\bar{\mathfrak{a}}\bar{\mathfrak{a}}'})} H^*(\bar{\mathfrak{a}}'))$$

Now by construction we have canonical compatible isomorphisms $H^*(\bar{\mathfrak{a}}) \cong H^*(\mathfrak{a})$, $H^*(\bar{\mathfrak{a}}') \cong H^*(\mathfrak{a})$, and $H^*(C_{\bar{\mathfrak{a}}\bar{\mathfrak{a}}'}) \cong H^*(\mathfrak{a})$. Put $\mathfrak{c} = (\mathfrak{a} \xleftarrow{\mathfrak{a}} \mathfrak{a})$.

We then have the following commutative diagram

$$
\begin{array}{ccccc}
Z(H^*(\mathfrak{a})) & \longleftarrow & Z(H^*(\mathfrak{c})) & \longrightarrow & Z(H^*(\mathfrak{a})) \\
\| & & \| & & \| \\
Z(H^*(\bar{\mathfrak{a}})) & \longleftarrow & Z(H^*(\bar{\mathfrak{c}})) & \longrightarrow & Z(H^*(\bar{\mathfrak{a}}')) \\
\sigma_{\bar{\mathfrak{a}}} \uparrow & & \sigma_{\bar{\mathfrak{c}}} \uparrow & & \sigma_{\bar{\mathfrak{a}}'} \uparrow \\
H\mathbf{C}^*(\bar{\mathfrak{a}}) & \xleftarrow{\cong} & H\mathbf{C}^*(\bar{\mathfrak{c}}) & \xrightarrow{\cong} & H\mathbf{C}^*(\bar{\mathfrak{a}}')
\end{array}
$$

Using the definition of $\mathfrak{c}$ it is trivial to see that the topmost horizontal arrows are isomorphisms and compose to the identity isomorphism $Z(H^*(\mathfrak{a})) \longrightarrow Z(H^*(\mathfrak{a}))$. In this way we obtain a commutative diagram

(4.4)
$$
\begin{array}{ccc}
Z(H^*(\mathfrak{a})) & = & Z(H^*(\mathfrak{a})) \\
\bar{\sigma}_{\mathfrak{a}} \uparrow & & \uparrow \bar{\sigma}_{\mathfrak{a}'} \\
H\mathbf{C}^*(\bar{\mathfrak{a}}) & \xrightarrow{\cong} & H\mathbf{C}^*(\bar{\mathfrak{a}}')
\end{array}
$$

where the bottom horizontal arrow is coming from the canonical $\text{Ho}(B_\infty)$ isomorphism between $\mathbf{C}^*(\bar{\mathfrak{a}})$ and $\mathbf{C}^*(\bar{\mathfrak{a}}')$. We now put $\sigma_{\mathfrak{a}} = \sigma_{\bar{\mathfrak{a}}}$. Diagram (4.4) shows that $\sigma_{\mathfrak{a}}$ is indeed well-defined in the appropriate sense.

5. Grothendieck categories

It follows from the definition (3.1) that we need to be able to understand the Hochschild cohomology of the category of injectives in a Grothendieck category. In this section we will prove the relevant technical results.

5.1. **A model structure.** We assume troughout that $\mathcal{C}$ is a k-linear Grothendieck category and as usual $C(\mathcal{C})$ denotes the category of complexes over $\mathcal{C}$. In this section we construct a generalization of the usual injective model structure on $C(\mathcal{C})$ [1, 6, 11]. It is used for some of the proofs below, but *not* for the statement of the results.

For a small k-DG-category $\mathfrak{a}$ consider the category $\mathsf{Dif}(\mathfrak{a}, \mathcal{C}) = \mathsf{DGFun}(\mathfrak{a}^{\mathrm{op}}, C(\mathcal{C}))$.

Proposition 5.1.   (1) $\mathsf{Dif}(\mathfrak{a}, \mathcal{C})$ *has the structure of a model category in which the weak equivalences are the pointwise quasi-isomorphisms and the cofibrations are the pointwise monomorphisms.*
   (2) *Suppose $\mathfrak{a}$ is k-cofibrant. If $F \in \mathsf{Dif}(\mathfrak{a}, \mathcal{C})$ is fibrant, then so is every $F(A)$ in $C(\mathcal{C})$.*

If $\mathfrak{a} = k$ then we obtain the usual injective model structure on $C(\mathcal{C})$. A very efficient proof for the existence of the latter has been given in [6] by Beke. Our proof of Proposition 5.1 is based on the following result which is an abstraction of the method used for $C(\mathcal{C})$ by Beke.

Proposition 5.2. [6] *Let $(H^i)_{i \in \mathbb{Z}} : \mathcal{A} \longrightarrow \mathcal{B}$ be additive functors between Grothendieck categories such that*

   (1) H^i *preserves filtered colimits;*
   (2) $(H^i)_i$ *is effaceable (i.e. every $A \in \mathcal{A}$ admits a mono $m : A \longrightarrow A'$ with $H^i(m) = 0$ for every $i \in \mathbb{Z}$, or, equivalently, $H^i(\mathsf{Inj}(\mathcal{A})) = 0$ for every $i \in \mathbb{Z}$);*
   (3) $(H^i)_i$ *is cohomological (i.e. for every short exact $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is $\mathcal{A}$ there is a long exact $\ldots \longrightarrow H^i(A') \longrightarrow H^i(A) \longrightarrow H^i(A'') \longrightarrow H^{i+1}(A') \longrightarrow \ldots$ in $\mathcal{B}$)*

Let $\mathsf{iso}_{\mathcal{B}}$ be the class of isomorphisms in $\mathcal{B}$ and $\mathsf{mono}_{\mathcal{A}}$ the class of monomorphisms in $\mathcal{A}$. There is a model structure on $\mathcal{A}$ such that

   (1) $\mathsf{mono}_{\mathcal{A}}$ *is the class of cofibrations;*
   (2) $\cap_{i \in \mathbb{Z}}(H^i)^{-1}(\mathsf{iso}_{\mathcal{B}})$ *is the class of weak equivalences.*

Proof. Our notations in this proof are the ones used in [6]. We use [6, Theorem 1.7] which is attributed to Jeffrey Smith. By [6, Proposition 1.12], $\mathsf{mono}_{\mathcal{A}} = \mathsf{cof}(I)$ for some set $I \subset \mathsf{mono}_{\mathcal{A}}$. We check the hypotheses for Theorem [6, Theorem 1.7] with $W = \cap_{i \in \mathbb{Z}}(H^i)^{-1}(\mathsf{iso}_{\mathcal{B}})$. (c0) and (c3) are automatic (using [6, Proposition 1.18]). For (c2), we are to show that $\mathsf{inj}(\mathsf{mono}_{\mathcal{A}}) \subset W$. So consider $f : X \longrightarrow Y$ in $\mathsf{inj}(\mathsf{mono}_{\mathcal{A}})$. It is easily seen using the lifting property of f that f is a split epimorphism with an injective kernel K. The result follows from the long exact sequence associated to $0 \longrightarrow K \longrightarrow X \longrightarrow Y \longrightarrow 0$ in which each $H^i(K) = 0$. For (c2), $\mathsf{mono}_{\mathcal{A}} \cap \cap_{i \in \mathbb{Z}}(H^i)^{-1}(\mathsf{iso}_{\mathcal{B}})$ is easily seen to be closed under pushouts using the long exact sequence, and under transfinite composition using that filtered colimits are exact in $\mathcal{A}$ and that H^i preserves them. $\square$

Proof of Proposition 5.1. The forgetful functor

$$\mathsf{Dif}(\mathfrak{a}, C(\mathcal{C})) \longrightarrow C(\mathcal{C}) : F \longmapsto F(A)$$

has a left adjoint

$$L_A : C(\mathcal{C}) \longrightarrow \mathsf{Dif}(\mathfrak{a}, C(\mathcal{C})) : C \longmapsto (A' \longmapsto \mathfrak{a}(A, A') \otimes_k C).$$

Since every $\mathfrak{a}(A, A')$ has projective components, L_A preserves monomorphisms. Since $\mathfrak{a}(A, A') \otimes -$ preserves acyclic complexes L_A preserves weak equivalences and hence L_A preserves trivial cofibrations. It now easily follows that if F has the lifting property with respect to trivial cofibrations then so does every $F(A)$.

Hence (2) follows if we prove (1). For (1), we use Proposition 5.2 for

$$H^i : \mathsf{Dif}(\mathfrak{a}, C(\mathcal{C})) \longrightarrow \mathcal{C}^{|\mathfrak{a}|} : M \longmapsto (H^i M(A))_A.$$

To see that $(H^i)_i$ is effaceable, we can take for $A \in \mathsf{Dif}(\mathfrak{a}, \mathcal{C})$ the monomorphism $A \longrightarrow \mathrm{cone}(1_A)$. $\qquad\square$

5.2. The derived Gabriel-Popescu theorem. The results in this section are probably well-known but we have not been able to locate a reference.

Let $j : \mathfrak{u} \longrightarrow \mathcal{C}$ be a k-linear functor from a small k-linear category inducing a *localization*

$$(5.1) \qquad\qquad c : \mathcal{C} \longrightarrow \mathsf{Mod}(\mathfrak{u}) : C \longmapsto \mathcal{C}(j(-), C)$$

By this we mean that c is fully faithful and has an exact left adjoint. By the Gabriel-Popescu theorem [32] c is a localization if j is fully faithful and generating. However in our applications to ringed spaces j will *not* be fully faithful. Necessary and sufficient conditions for c to be a localization were given in [29].

We start with the following easy result.

Theorem 5.2.1. *The functor*

$$D(\mathcal{C}) \longrightarrow D(\mathfrak{u})$$

which sends a fibrant object $A \in C(\mathcal{C})$ to $\mathrm{Hom}_{\mathcal{C}}(j(-), A)$ preserves RHom.

Proof. We need to prove that for fibrant A, B we have a quasi-isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{RHom}_{\mathfrak{u}}(\mathrm{Hom}_{\mathcal{C}}(j(-), A), \mathrm{Hom}_{\mathcal{C}}(j(-), B))$$

Since c is fully faithful we have

$$\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathfrak{u}}(\mathrm{Hom}_{\mathcal{C}}(j(-), A), \mathrm{Hom}_{\mathcal{C}}(j(-), B))$$

And since c has an exact left adjoint we easily deduce that $\mathrm{Hom}_{\mathfrak{u}}(-, \mathrm{Hom}_{\mathcal{C}}(j(-), B))$ preserves acyclic complexes. Thus

$$\mathrm{Hom}_{\mathfrak{u}}(\mathrm{Hom}_{\mathcal{C}}(j(-), A), \mathrm{Hom}_{\mathcal{C}}(j(-), B)) = \mathrm{RHom}_{\mathfrak{u}}(\mathrm{Hom}_{\mathcal{C}}(j(-), A), \mathrm{Hom}_{\mathcal{C}}(j(-), B))$$

This finishes the proof. $\qquad\square$

Now we discuss a more sophisticated derived version of the Gabriel-Popescu theorem. Let $l : \mathfrak{f} \longrightarrow C(\mathcal{C})$ be any fully faithful DG-functor such that:

(1) $l(\mathfrak{f})$ consists of fibrant complexes;
(2) every object in $j(\mathfrak{u})$ is quasi-isomorphic to an object in $l(\mathfrak{f})$;
(3) the only cohomology of an object in $l(\mathfrak{f})$ is in degree zero and lies in $j(\mathfrak{u})$.

For example $\mathfrak{f}$ could consist of injective resolutions for the objects $j(U)$.

Theorem 5.2.2. *The functor*

$$D(\mathcal{C}) \longrightarrow D(\mathfrak{f})$$

which sends a fibrant object $A \in C(\mathcal{C})$ to $\mathrm{Hom}_{\mathcal{C}}(l(-), A)$ preserves RHom.

The rest of this section will be devoted to the proof of this theorem. Along the way we introduce some notations which will also be used afterwards.

To start we note the following.

Lemma 5.2.3. *It is sufficient to prove Theorem 5.2.2 for one particular choice of $\mathfrak{f}$.*

Proof. Let $\mathfrak{f}_1$ be the full DG-subcategory of $C(\mathcal{C})$ of all fibrant complexes satisfying (3). Then $\mathfrak{f} \longrightarrow \mathfrak{f}_1$ is a quasi-equivalence, hence the corresponding functor $\mathsf{Dif}(\mathfrak{f}_1) \longrightarrow \mathsf{Dif}(\mathfrak{f})$ preserves RHom (Lemma 2.2.2). Therefore, if Theorem 5.2.2 is true for one $\mathfrak{f}$, it is true for $\mathfrak{f}_1$ and then it is true for all $\mathfrak{f}$. $\qquad\square$

Ideally we would want to choose $\mathfrak{f}$ in such a way that there is a corresponding DG-functor $\mathfrak{u} \longrightarrow \mathfrak{f}$. It is not clear to us that this is possible in general. However, as we now show, it is possible after replacing $\mathfrak{u}$ by a k-cofibrant resolution.

Let $r : \overline{\mathfrak{u}} \longrightarrow \mathfrak{u}$ be a k-cofibrant resolution of $\mathfrak{u}$. Let $i : \mathcal{C} \longrightarrow C(\mathcal{C})$ be the canonical inclusion and let $ijr \longrightarrow E$ be a fibrant resolution for the model-structure on $\mathsf{Dif}(\overline{\mathfrak{u}}^{\mathrm{op}}, \mathcal{C})$ of §5.1. By Proposition 5.1 this yields fibrant replacements $ij(U) \longrightarrow E(U)$ natural in $U \in \overline{\mathfrak{u}}$. We define a new DG-category $\overline{\overline{\mathfrak{u}}}$ with the same objects as $\mathfrak{u}$ and

$$\overline{\overline{\mathfrak{u}}}(U, V) = \mathrm{Hom}_{\mathcal{C}}(E(U), E(V))$$

By construction we have the following commutative diagram of DG-functors

$$
\begin{array}{ccc}
\mathfrak{u} & \xrightarrow{\;ij\;} & C(\mathcal{C}) \\[4pt]
{\scriptstyle r}\big\uparrow & & \big\uparrow{\scriptstyle E(-)} \\[4pt]
\overline{\mathfrak{u}} & \xrightarrow[\;f\;]{} & \overline{\overline{\mathfrak{u}}}
\end{array}
$$

where r is a quasi-equivalence and $E(-)$ is fully faithful.

Proof of Theorem 5.2.2. We will prove the theorem with $\mathfrak{f} = \overline{\overline{\mathfrak{u}}}$.

Step 1. Let $A \in C(\mathcal{C})$ be fibrant. We first claim that the canonical map

$$\mathrm{Hom}_{\mathcal{C}}(E(-), A) \longrightarrow Rf^! f_* \mathrm{Hom}_{\mathcal{C}}(E(-), A)$$

is a quasi-isomorphism in $\mathsf{Dif}(\overline{\overline{\mathfrak{u}}})$. The proof is based on the following computation for $U \in \mathrm{Ob}(\mathfrak{u})$

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(E(U), A) &\cong \mathrm{RHom}_{\mathfrak{u}}(\mathrm{Hom}_{\mathcal{C}}(j(-), E(U)), \mathrm{Hom}_{\mathcal{C}}(j(-), A)) \\
&\cong \mathrm{RHom}_{\overline{\mathfrak{u}}}(\mathrm{Hom}_{\mathcal{C}}(jr(-), E(U)), \mathrm{Hom}_{\mathcal{C}}(jr(-), A)) \\
&\cong \mathrm{RHom}_{\overline{\mathfrak{u}}}(\overline{\overline{\mathfrak{u}}}(-, U), \mathrm{Hom}_{\mathcal{C}}(E(-), A)) \\
&= (Rf^! f_* \mathrm{Hom}_{\mathcal{C}}(E(-), A))(U)
\end{aligned}
$$

The first line is Theorem 5.2.1. In the second line we use the fact that $\overline{\mathfrak{u}} \longrightarrow \mathfrak{u}$ is a quasi-equivalence together with Lemma 2.2.2. The third line is a change of notation and the fourth line is an easy verification.

Step 2. Now we finish the proof of the theorem. We compute for fibrant $A, B \in C(\mathcal{C})$

$$\mathrm{RHom}_{\bar{\bar{\mathfrak{u}}}}(\mathrm{Hom}_{\mathcal{C}}(E(-), A), \mathrm{Hom}_{\mathcal{C}}(E(-), B))$$

$$\cong \mathrm{RHom}_{\bar{\bar{\mathfrak{u}}}}(\mathrm{Hom}_{\mathcal{C}}(E(-), A), Rf^! f_* \mathrm{Hom}_{\mathcal{C}}(E(-), B))$$

$$\cong \mathrm{RHom}_{\bar{\mathfrak{u}}}(f_* \mathrm{Hom}_{\mathcal{C}}(E(-), A), f_* \mathrm{Hom}_{\mathcal{C}}(E(-), B))$$

$$\cong \mathrm{RHom}_{\bar{\mathfrak{u}}}(\mathrm{Hom}_{\mathcal{C}}(j(-), A), \mathrm{Hom}_{\mathcal{C}}(j(-), B))$$

$$\cong \mathrm{RHom}_{\mathfrak{u}}(\mathrm{Hom}_{\mathcal{C}}(j(-), A), \mathrm{Hom}_{\mathcal{C}}(j(-), B))$$

$$= \mathrm{Hom}_{\mathcal{C}}(A, B)$$

where we have once again used Theorem 5.2.1 and the fact that $\bar{\mathfrak{u}} \longrightarrow \mathfrak{u}$ is a quasi-equivalence. $\qquad\square$

Remark 5.2.4. If $\mathfrak{u}$ is already k-cofibrant then we may take $\bar{\mathfrak{u}} = \mathfrak{u}$. By letting E be an injective resolution of ij we obtain injective resolutions $U \longrightarrow E(U)$ of $j(U)$ natural in U.

5.3. **Hochschild complexes.** Let $\mathfrak{i} = \mathsf{Inj}(\mathcal{C})$ be the category of injectives in $\mathcal{C}$ and let $l : \mathfrak{f} \longrightarrow C(\mathcal{C})$ be as in §5.2. In this section we prove the following comparison result.

Theorem 5.3.1. *There is a quasi-isomorphism* $\mathbf{C}_{\mathrm{sh}}(\mathfrak{i}) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{f})$.

Corollary 5.3.2. $\mathbf{C}_{\mathrm{sh}}(\mathfrak{i})$ *has small cohomology.*

Proof. It is clear that we may take $\mathfrak{f}$ to be small. $\qquad\square$

Proof of Theorem 5.3.1. We define the $\mathfrak{i}-\mathfrak{f}$-bimodule

$$X(U, E) = \mathrm{Hom}_{\mathcal{C}}(l(U), E).$$

By Lemma 2.2.2 and the derived Gabriel-Popescu Theorem (Theorem 5.2.2)

$$\mathrm{RHom}_{\mathfrak{f}}(X(-, E), X(-, F)) \cong \mathrm{Hom}_{\mathcal{C}}(E, F)$$

$$\cong \mathfrak{i}(E, F)$$

On the other hand,

$$\mathrm{RHom}_{\mathfrak{i}^{\mathrm{op}}}(X(V, -), X(U, -)) \cong \mathrm{RHom}_{\mathfrak{i}^{\mathrm{op}}}(\mathcal{C}(l(V), -), \mathcal{C}(l(U), -))$$

$$\cong \mathrm{RHom}_{\mathcal{C}}(l(U), l(V))$$

$$\cong \mathfrak{f}(U, V)$$

where the first line is a consequence of Lemma 5.3.3 below. The result now follows from Proposition 4.3.3. $\qquad\square$

Lemma 5.3.3. *Suppose a small abelian category* $\mathcal{A}$ *has enough injectives in* $\mathsf{add}(\mathfrak{j})$ *for* $\mathfrak{j} \subset \mathsf{Inj}(\mathcal{A})$ *(i.e., for every object in* $\mathcal{A}$ *there is a mono into a finite sum of injectives in* $\mathfrak{j}$*). Consider* $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathsf{Mod}(\mathfrak{j}) : A \longmapsto \mathcal{A}(A, -)$. *For* $A, B \in \mathcal{A}$, *we have*

$$\mathrm{RHom}_{\mathcal{A}}(A, B) \cong \mathrm{RHom}_{\mathfrak{j}^{\mathrm{op}}}(\mathcal{A}(B, -), \mathcal{A}(A, -)).$$

Proof. Let $B \longrightarrow I^{\cdot}$ be an injective resolution of B in $\mathsf{add}(\mathfrak{j})$. Then $\mathcal{A}(I^{\cdot}, -) \longrightarrow \mathcal{A}(B, -)$ is a resolution in $\mathsf{Mod}(\mathfrak{j})$, and every object $\mathcal{A}(I^i, -) = \mathcal{A}(\oplus_{k=1}^n J_k^i, -) = \oplus_{k=1}^n \mathfrak{j}(J_k^i, -)$ is projective. $\qquad\square$

In the proof of Theorem 5.3.1 we have used this lemma in the case $\mathfrak{j} = \mathsf{Inj}(\mathcal{A})$. The added generality will be used in §7.7.

5.4. **A spectral sequence.** We keep the same notations as in §5.2. The main theorem of this section is an interesting spectral sequence which relates the Hochschild cohomology of $\mathfrak{u}$ to that of $\mathfrak{i} = \mathsf{Inj}(\mathcal{C})$.

Theorem 5.4.1. *There is a convergent, first quadrant spectral sequence*

$$(5.2) \qquad E_2^{pq} : \mathrm{HC}_{\mathrm{sh}}^p(\mathfrak{u}, \mathrm{Ext}_{\mathcal{C}}^q(j(-), j(-))) \Rightarrow \mathrm{HC}_{\mathrm{sh}}^{p+q}(\mathfrak{i})$$

The proof of Theorem 5.4.1 depends on the following technical result.

Lemma 5.4.2. *There is a (non B_∞-)quasi-isomorphism* $\mathbf{C}(\bar{\mathfrak{u}}, \bar{\bar{\mathfrak{u}}}) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{i})$.

Proof. By Theorem 5.3.1 it is sufficient to construct a quasi-isomorphism $\mathbf{C}(\bar{\mathfrak{u}}, \bar{\bar{\mathfrak{u}}}) \cong \mathbf{C}_{\mathrm{sh}}(\bar{\bar{\mathfrak{u}}})$. Let $\mathfrak{v}$ be a cofibrant resolution of $\bar{\bar{\mathfrak{u}}}$ as a $\bar{\bar{\mathfrak{u}}}-\bar{\mathfrak{u}}$-bimodule.

Of course

$$(5.3) \qquad \mathrm{RHom}_{\bar{\bar{\mathfrak{u}}}^{\mathrm{op}}}(\mathfrak{v}(V,-), \mathfrak{v}(U,-)) \cong \mathrm{RHom}_{\bar{\bar{\mathfrak{u}}}^{\mathrm{op}}}(\bar{\bar{\mathfrak{u}}}(V,-), \bar{\bar{\mathfrak{u}}}(U,-)) \cong \bar{\bar{\mathfrak{u}}}(U,V)$$

so we compute

$$\begin{aligned}
\mathrm{RHom}_{\bar{\mathfrak{u}}}(\mathfrak{v}(-,U), \mathfrak{v}(-,V)) &\cong \mathrm{RHom}_{\bar{\mathfrak{u}}}(\bar{\bar{\mathfrak{u}}}(-,U), \bar{\bar{\mathfrak{u}}}(-,V)) \\
&\cong \mathrm{RHom}_{\bar{\mathfrak{u}}}(\mathrm{Hom}_{\mathcal{C}}(jr(-), E(U)), \mathrm{Hom}_{\mathcal{C}}(jr(-), E(V))) \\
&\cong \mathrm{RHom}_{\mathfrak{u}}(\mathrm{Hom}_{\mathcal{C}}(j(-), E(U)), \mathrm{Hom}_{\mathcal{C}}(j(-), E(V))) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(E(U), E(V)) \\
&\cong \bar{\bar{\mathfrak{u}}}(U,V)
\end{aligned}$$

where we have use that $\bar{\mathfrak{u}} \longrightarrow \mathfrak{u}$ is a quasi-isomorphism, together with Theorem 5.2.1. The result now follows from Theorem 4.1.1.1. $\qquad\square$

Proof or Theorem 5.4.1. We use Lemma 5.4.2. Let $\bar{\mathfrak{u}}$ be a semi-free [10, §13.4] resolution of $\mathfrak{u}$. This is in particular a k-cofibrant resolution $\bar{\mathfrak{u}} \longrightarrow \mathfrak{u}$ concentrated in non-positive degree. The latter implies that the truncations $\tau_{\leq n}\bar{\bar{\mathfrak{u}}}$ and $\tau_{\geq n}\bar{\bar{\mathfrak{u}}} = \bar{\bar{\mathfrak{u}}}/\tau_{<n}\bar{\bar{\mathfrak{u}}}$ are $\bar{\mathfrak{u}}$-bimodules. Recall that the homology of $\bar{\bar{\mathfrak{u}}}$ is $\mathrm{Ext}_{\mathcal{C}}^*(j(-), j(-))$ so it lives in non-negative degree. So up to quasi-isomorphism we may replace $\bar{\bar{\mathfrak{u}}}$ by $\mathfrak{w} = \tau_{\geq 0}\bar{\bar{\mathfrak{u}}}$. We put the ascending filtration $(\tau_{\leq n}\mathfrak{w})_n$ on $\mathfrak{w}$. This filtration is positive since $\tau_{<0}\mathfrak{w} = 0$.

We claim that the obvious map

$$\bigcup_n \mathbf{C}(\bar{\mathfrak{u}}, \tau_{\leq n}\mathfrak{w}) \longrightarrow \mathbf{C}(\bar{\mathfrak{u}}, \mathfrak{w})$$

is a quasi-isomorphism. To prove this it is sufficient to show that for a fixed i the map

$$\mathrm{HC}^i(\bar{\mathfrak{u}}, \tau_{\leq n}\mathfrak{w}) \longrightarrow \mathrm{HC}^i(\bar{\mathfrak{u}}, \mathfrak{w})$$

is a quasi-isomorphism for large n. Equivalently by the long exact sequence for Hochschild cohomology, $\mathrm{HC}^i(\bar{\mathfrak{u}}, \tau_{\geq n}\mathfrak{w})$ should be zero for large n. Now by (2.5) we have

$$\mathrm{HC}^i(\bar{\mathfrak{u}}, \tau_{\geq n}\mathfrak{w}) = \mathrm{RHom}_{\bar{\mathfrak{u}}^{\mathrm{op}} \otimes \bar{\mathfrak{u}}}(\bar{\mathfrak{u}}, \tau_{\geq n}\mathfrak{w}[i])$$

Since a cofibrant $\bar{\mathfrak{u}}^{\mathrm{op}} \otimes \bar{\mathfrak{u}}$-resolution of $\bar{\mathfrak{u}}$ may also be chosen to live in non-positive degree it is clear that this is zero for $n > i$.

So the spectral sequence associated to the filtered complex $\bigcup_n \mathbf{C}(\bar{\mathfrak{u}}, \tau_{\leq n}\mathfrak{w})$ converges to $\mathrm{HC}_*^*(\mathfrak{i})$. The associated graded complex is

$$\bigoplus_n \mathbf{C}(\bar{\mathfrak{u}}, \mathrm{Ext}_{\mathcal{C}}^n(j(-), j(-))[-n])$$

and the homology of this graded complex is:

$$\bigoplus_{mn} \mathrm{HC}_{\mathrm{sh}}^{m-n}(\mathfrak{u}, \mathrm{Ext}_{\mathcal{C}}^n(j(-), j(-)))$$

After the appropriate reindexing we obtain the desired result. $\square$

5.5. **Application of a censoring relation.** Lemma 5.4.2 has the following useful variant:

Proposition 5.5.1. *Assume that $\mathfrak{u}$ is equipped with a (possibly trivial) censoring relation $\mathcal{R}$ (see §4.3) such that*

$$\mathrm{Ext}_{\mathcal{C}}^i(j(U), j(V)) = 0 \text{ for } i > 0 \text{ and } (U, V) \in \mathcal{R}$$

Then there is a quasi-isomorphism

$$\mathbf{C}_{\mathrm{ab}}(\mathcal{C}) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{u})$$

Proof. We define the $\mathfrak{i}-\mathfrak{u}$-bimodule

$$X(U, E) = \mathrm{Hom}_{\mathcal{C}}(j(U), E).$$

By Theorem 5.2.1

$$\mathrm{RHom}_{\mathfrak{u}}(X(-, E), X(-, F)) \cong \mathrm{Hom}_{\mathcal{C}}(E, F)$$
$$\cong \mathfrak{i}(E, F)$$

On the other hand for $(U, V) \in \mathcal{R}$

$$\mathrm{RHom}_{\mathfrak{i}^{\mathrm{op}}}(X(V, -), X(U, -)) \cong \mathrm{RHom}_{\mathfrak{i}^{\mathrm{op}}}(\mathcal{C}(j(V), -), \mathcal{C}(j(U), -))$$
$$\cong \mathrm{RHom}_{\mathcal{C}}(j(U), j(V))$$
$$\cong \mathfrak{u}(U, V)$$

where the first line is a consequence of Lemma 5.3.3. The result now follows from Proposition 4.3.3. $\square$

6. Basic results about Hochschild cohomology of abelian categories

Let $\mathcal{A}$ be a small abelian category. By definition we have $\mathbf{C}_{\mathrm{ab}}(\mathcal{A}) = \mathbf{C}_{\mathrm{sh}}(\mathfrak{i})$ with $\mathfrak{i} = \mathrm{Inj}\,\mathrm{Ind}(\mathfrak{a})$ (see §3). The embedding $\mathcal{A} \longrightarrow \mathrm{Ind}(\mathcal{A})$ satisfies the hypotheses on j in §5 so the results of that section apply. We will now translate them to the current setting.

The first result below relates the Hochschild cohomology of $\mathcal{A}$ to that of suitable small DG-categories. Let ${}^e\mathcal{A}$ be the full DG-subcategory of $C(\mathrm{Ind}(\mathcal{A}))$ spanned by all positively graded complexes of injectives whose only cohomology is in degree zero and lies in $\mathcal{A}$ and let ${}^eD^b(\mathcal{A})$ be spanned by all left bounded complexes of injectives with bounded cohomology in $\mathcal{A}$. Note that for example by [30, Prop. 2.14] we have

$$H^0({}^eD^b(\mathcal{A})) \cong D^b(\mathcal{A})$$

so ${}^eD^b(\mathcal{A})$ is a DG-enhancent for $D^b(\mathcal{A})$.

Theorem 6.1. *There are quasi-isomorphisms*

$$\mathbf{C}_{\mathrm{ab}}(\mathcal{A}) \cong \mathbf{C}_{\mathrm{sh}}({}^e\mathcal{A}) \cong \mathbf{C}_{\mathrm{sh}}({}^eD^b(\mathcal{A}))$$

Proof. The first quasi-isomorphism follows from Theorem 5.3.1. For the second quasi-isomorphism put $\mathfrak{a} = {}^e\mathcal{A}$ and let $\mathfrak{b}$ be the closure of $\mathfrak{a}$ in $C(\mathsf{Ind}\mathcal{A})$ under finite cones and shifts. Then $\mathfrak{b} \longrightarrow {}^e D^b(\mathcal{A})$ is a quasi-equivalence. Furthermore the functor $B \longmapsto \mathrm{Hom}_{C(\mathsf{Ind}\mathcal{A})}(-, B)$ defines an embedding $\mathfrak{b} \longrightarrow \mathsf{Dif}(\mathfrak{a})$ whose image consists of cofibrant objects. We may now invoke Theorem 4.4.1. □

In [19], Bernhard Keller defines the Hochschild complex $\mathbf{C}_{\mathrm{ex}}(\mathcal{E})$ of an exact category $\mathcal{E}$ as $\mathbf{C}_{\mathrm{sh}}(\mathcal{Q})$ for a DG-quotient $\mathcal{Q}$ of $Ac^b(\mathcal{E}) \longrightarrow C^b(\mathcal{E})$, where $C^b(\mathcal{E})$ is the DG-category of bounded complexes of $\mathcal{E}$-objects and $Ac^b(\mathcal{E})$ is its full DG-subcategory of acyclic complexes. We endow the abelian category $\mathcal{A}$ with the exact structure given by all exact sequences.

Theorem 6.2. *There is a quasi-isomorphism*

$$\mathbf{C}_{\mathrm{ab}}(\mathcal{A}) \cong \mathbf{C}_{\mathrm{ex}}(\mathcal{A}).$$

Proof. This follows from Theorem 6.1 and Lemma 6.3 below. □

Lemma 6.3. ${}^e D^b(\mathcal{A})$ *is a DG-quotient of* $Ac^b(\mathcal{A}) \longrightarrow C^b(\mathcal{A})$.

Proof. We sketch the proof. Let $\mathfrak{c}$ be the following DG-category: the objects of $\mathfrak{c}$ are quasi-isomorphisms $f : C \longrightarrow I$ with $C \in C^b(\mathcal{A})$ and $I \in {}^e D^b(\mathcal{A})$. Morphisms from f to $g : D \longrightarrow J$ are maps $\mathrm{cone}(f) \longrightarrow \mathrm{cone}(g)$ with zero component $I \longrightarrow D[1]$. The two projections yield a diagram $C^b(\mathcal{A}) \longleftarrow \mathfrak{c} \longrightarrow {}^e D^b(\mathcal{A})$, for which it is easily seen that $C^b(\mathcal{A}) \longleftarrow \mathfrak{c}$ is a quasi-equivalence and $\mathfrak{c} \longrightarrow {}^e D^b(\mathcal{A})$ induces the exact sequence of associated triangulated categories $H^0(Ac^b(\mathcal{A})) \longrightarrow H^0(C^b(\mathcal{A})) \longrightarrow D^b_{\mathcal{A}}(\mathsf{Ind}(\mathcal{A})) \cong D^b(\mathcal{A})$. By [22, 10], this proves the statement. □

Remark 6.4. Since $\mathbf{C}_{\mathrm{ex}}$ is easily seen to satisfy $\mathbf{C}_{\mathrm{ex}}(\mathcal{E}) \cong \mathbf{C}_{\mathrm{ex}}(\mathcal{E}^{\mathrm{op}})$ (in $D(k)$) if $\mathcal{E}^{\mathrm{op}}$ is endowed with the opposite exact sequences of $\mathcal{E}$, by Theorem 6.2 we have in particular that $\mathbf{C}_{\mathrm{ab}}(\mathcal{A}) \cong \mathbf{C}_{\mathrm{ab}}(\mathcal{A}^{\mathrm{op}})$. It is a pleasant excercise to derive this result directly from our definition.

The following result, of theoretical interest, is a restatement of a special case of the spectral sequence (5.2). It compares the Hochschild cohomology of $\mathcal{A}$ as linear and as abelian category.

Proposition 6.5. *There is a convergent, first quadrant spectral sequence*

$$(6.1) \qquad E_2^{pq} : \mathrm{HC}_{\mathrm{sh}}^p(\mathcal{A}, \mathrm{Ext}_{\mathcal{A}}^q(-, -)) \Rightarrow \mathrm{HC}_{\mathrm{ab}}^{p+q}(\mathcal{A})$$

Proof. We only need to remark that Yoneda-Ext computed in $\mathcal{A}$ and $\mathsf{Ind}\mathcal{A}$ is the same. See for example [30, Prop. 2.14]. □

The following result shows that if $\mathcal{A}$ has enough injectives then there is no need to pass to $\mathsf{Ind}\mathcal{A}$.

Theorem 6.6. *Assume that $\mathcal{A}$ has enough injectives and put $\mathfrak{i} = \mathsf{Inj}\mathcal{A}$. There is a quasi-isomorphism*

$$\mathbf{C}_{\mathrm{ab}}(\mathcal{A}) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{i}).$$

Proof. Let $\mathfrak{k}$ be the the full DG-subcategory of $C(\mathcal{A})$ spanned by all positively graded complexes of $\mathfrak{i}$-objects whose only cohomology is in degree zero. Then the inclusion $\mathfrak{k} \longrightarrow {}^e\mathcal{A}$ is a quasi-equivalence so using Theorem 6.1 it is sufficient to show that $\mathfrak{k}$ and $\mathfrak{i}$ have isomorphic Hochschild complexes. We embed $\mathfrak{k}$ in $\mathsf{Dif}(\mathfrak{i})$ via the functor $E \mapsto \mathrm{Hom}_{\mathcal{C}}(-, E)$. Then $\mathfrak{k}$ is mapped to right bounded projective

complexes. Such complexes are cofibrant and hence we may use Theorem 4.4.1 to deduce $\mathbf{C}_{\mathrm{sh}}(\mathfrak{k}) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{i})$. □

The following corollary will be used in §7.7.

Corollary 6.7. *Assume that $\mathcal{A}$ has enough injectives in* $\mathsf{add}(\mathfrak{j})$*. There is a quasi-isomorphism*
$$\mathbf{C}_{\mathrm{ab}}(\mathcal{A}) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{j}).$$

Proof. Put $\mathfrak{i} = \mathsf{Inj}(\mathcal{A})$. By Lemma 5.3.3, the inclusion $\mathfrak{j} \longrightarrow \mathfrak{i}$ satisfies the hypotheses of Proposition 4.3.4, hence $\mathbf{C}_{\mathrm{sh}}(\mathfrak{i}) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{j})$. The result now follows from Theorem 6.6. □

Corollary 6.8. *If $\mathcal{A}$ is a small abelian category then there is a quasi-isomorphism*
$$\mathbf{C}_{\mathrm{ab}}(\mathcal{A}) \cong \mathbf{C}_{\mathrm{ab}}(\mathsf{Ind}(\mathcal{A})).$$

Proof. Immediate from Theorem 6.6 and the definition. □

Theorem 6.6 applies in particular if $\mathcal{A}$ is a Grothendieck category. So the results in §5 (with $\mathcal{C}$ replaced by $\mathcal{A}$) may be reinterpreted as being about the Hochschild complex of a Grothendieck category. We mention in particular Theorem 5.3.1 and Proposition 5.5.1 which shows how to compute $\mathbf{C}_{\mathrm{ab}}(\mathcal{A})$ in terms of generators, Corollary 5.3.2 which shows that $\mathbf{C}_{\mathrm{ab}}(\mathcal{A})$ has small homology and the spectral sequence (5.2) abutting to $H\mathbf{C}_{\mathrm{ab}}^*(\mathcal{A})$.

The following corollary to Proposition 5.5.1 was our original motivation for starting this project.

Corollary 6.9. *Let $\mathfrak{a}$ be a small k-category. There is a quasi-isomorphism*
$$\mathbf{C}_{\mathrm{ab}}(\mathsf{Mod}(\mathfrak{a})) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{a}).$$

In particular, for a k-algebra A, there is a quasi-isomorphism
$$\mathbf{C}_{\mathrm{ab}}(\mathsf{Mod}(A)) \cong \mathbf{C}_{\mathrm{sh}}(A).$$

Proof. We apply Proposition 5.5.1 with the Yoneda embedding $j : \mathfrak{a} \longrightarrow \mathsf{Mod}(\mathfrak{a})$. □

The following proposition shows that $H\mathbf{C}_{\mathrm{ab}}^*(\mathcal{A})$ defines elements in the center of $D^b(\mathcal{A})$.

Proposition 6.10. *There is a homomorphism of graded rings*
$$\sigma_{\mathcal{A}} : H\mathbf{C}_{\mathrm{ab}}^*(\mathcal{A}) \longrightarrow Z(D^b(\mathcal{A}))$$
where on the right hand side we view $D^b(\mathcal{A})$ as a graded category in the usual way.

Proof. Put $\mathfrak{a} = {}^eD^b(\mathcal{A})$. Then by §4.5 there is a homomorphism of graded rings
$$\sigma_{\mathcal{A}} : H\mathbf{C}_{\mathrm{sh}}^*(\mathfrak{a}) \longrightarrow Z(H^*(\mathfrak{a}))$$
We define $\sigma_{\mathcal{A}}$ as the composition
$$H\mathbf{C}_{\mathrm{ab}}^*(\mathcal{A}) \cong H\mathbf{C}_{\mathrm{sh}}^*(\mathfrak{a}) \xrightarrow{\sigma_{\mathfrak{a}}} Z(H^*(\mathfrak{a})) \cong Z(D^b(\mathfrak{a}))$$
where the first isomorphism comes from Theorem 6.1. □

Remark 6.11. We may think of $\sigma_{\mathcal{A}}$ as defining "universal" elements in $\mathrm{Ext}_{\mathcal{A}}^*(M, M)$ for every $M \in D^b(\mathcal{A})$. These universal elements are closely related to Atiyah classes in algebraic geometry. See for example [9].

7. Hochschild cohomology for ringed spaces and schemes

7.1. **Discussion and statement of the main results.** Below let $(X, \mathcal{O})$ be a k-linear possibly non-commutative ringed space. We define the Hochschild complex of X as

$$\mathbf{C}(X) = \mathbf{C}_{\mathrm{ab}}(\mathsf{Mod}(X))$$

where $\mathsf{Mod}(X)$ is the category of sheaves of right modules over X. For the purpose of clarity we will sometimes use the notation $\mathbf{C}(X, \mathcal{O})$ for $\mathbf{C}(X)$. Note that in the definition of $\mathbf{C}(X)$ the bimodule structure of $\mathcal{O}$ does not enter explicitly.

As $\mathsf{Mod}(X)$ has enough injectives an equivalent definition (using Theorem 6.6) for $\mathbf{C}(X)$ would be

$$\mathbf{C}(X) = \mathbf{C}_{\mathrm{sh}}(\mathsf{InjMod}(X))$$

Recall that $\mathbf{C}(X)$ describes the deformation theory of the abelian category $\mathsf{Mod}(X)$ (as explained in §3) but *not* of the ringed space $(X, \mathcal{O})$. This is a related but different problem.[2]

We now summarize some of the results we will prove about $\mathbf{C}(X)$. We would like to think of $H\mathbf{C}^*(X)$ as defining a (generalized) cohomology theory for ringed spaces. A first indication for this is that $\mathbf{C}(-)$ is a contravariant functor on open embeddings of ringed spaces and associated to an open covering $X = U \cup V$ there is a corresponding Mayer-Vietoris long exact sequence (see §7.9)

(7.1)
$$\cdots \to H\mathbf{C}^{i-1}(U \cap V) \to H\mathbf{C}^i(X) \to H\mathbf{C}^i(U) \oplus H\mathbf{C}^i(V) \to H\mathbf{C}^i(U \cap V) \to \cdots$$

Let $\underline{k}$ be the constant sheaf with values in k. Our next interesting result is an isomorphism between Hochschild cohomology and ordinary cohomology

(7.2)
$$H\mathbf{C}^*(X, \underline{k}) \cong H^*(X, \underline{k}).$$

In [5] Baues shows that the singular cochain complex of a topological space is a B_∞-algebra. Thus (7.2) suggests that $\mathbf{C}(X, \underline{\mathbb{Z}})$ should be viewed as an algebraic analog of the singular cochain complex of X.

Now we discuss some more specific results for Hochschild cohomology. For any subposet $\mathcal{U}$ of $\mathsf{Open}(X)$ let $\mathfrak{u} = \mathfrak{u}(\mathcal{U})$ be the linear category with $\mathrm{Ob}(\mathfrak{u}) = \mathcal{U}$ and

$$\mathfrak{u}(U, V) = \begin{cases} \mathcal{O}(U) & \text{if } U \subset V \\ 0 & \text{otherwise} \end{cases}$$

First assume that $\mathcal{B}$ is a basis of X of *acyclic opens*, i.e. for $U \in \mathcal{B}$: $H^i(U, \mathcal{O}_U) = 0$ for $i > 0$. Put $\mathfrak{b} = \mathfrak{u}(\mathcal{B})$. Our first result (see §7.3) is that there is a quasi-isomorphism

(7.3)
$$\mathbf{C}(X) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{b}).$$

In [13, 15] Gerstenhaber and Schack define the *k-relative* Hochschild complex of a presheaf of rings. In order to make a connection with our setting let us assume that k is a field. Let $\mathcal{B}$ be an acyclic basis as above and let $\mathcal{O}_\mathcal{B}$ be the restriction of $\mathcal{O}$ to $\mathcal{B}$, considered as a presheaf of rings. It is implied in [13, 15] that

(7.4)
$$\mathbf{C}_{GS}(\mathcal{O}_\mathcal{B}) \stackrel{\mathrm{def}}{=} \mathrm{RHom}_{\mathcal{O}_\mathcal{B}^{\mathrm{op}} \otimes \mathcal{O}_\mathcal{B}}(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$$

---

[2]To obtain the correct correspondence we should deform $(X, \mathcal{O})$ in the category of algebroids over X, see [24]

is a reasonable definition for the Hochschild complex of X. We will show that this is true. Indeed it follows from combining (7.3) with (7.8)(7.9) below that

$$\mathbf{C}(X) \cong \mathbf{C}_{GS}(\mathcal{O}_{\mathcal{B}})$$

Let k be general again. We now specialize to the case where X is a quasi-compact separated scheme over k.

Let $X = \cup_{i=1}^{n} A_i$ be a finite affine open covering of X and let $\mathcal{A}$ be the closure of this covering under intersections. Put $\mathfrak{a} = \mathfrak{u}(\mathcal{A})$. Of course $\mathfrak{a}$ is not a basis for X but nevertheless we have the following analog of (7.3) (see §7.5)

$$(7.5) \qquad\qquad \mathbf{C}(X) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{a}).$$

Let $\mathsf{Qch}(X)$ be the category of quasi-coherent $\mathcal{O}$-modules. We will prove (see §7.7)

$$\mathbf{C}(X) \cong \mathbf{C}_{\mathrm{ab}}(\mathsf{Qch}(X)).$$

and if X is noetherian we even have

$$\mathbf{C}(X) \cong \mathbf{C}_{\mathrm{ab}}(\mathsf{coh}(X)).$$

where $\mathsf{coh}(X)$ is the category of coherent $\mathcal{O}$-modules.

If k is a field then in [39], Richard G. Swan defines the Hochschild complex of $(X, \mathcal{O})$ to be

$$\mathbf{C}_{\mathrm{Swan}}(X) \stackrel{\mathrm{def}}{=} \mathrm{RHom}_{X \times X}(\mathcal{O}_{\Delta}, \mathcal{O}_{\Delta})$$

where $\Delta \subset X \times X$ is the diagonal. We prove

$$(7.6) \qquad\qquad \mathbf{C}(X) \cong \mathbf{C}_{\mathrm{Swan}}(X)$$

This is known in the finite type case since in that case by [39, §3]

$$(7.7) \qquad\qquad \mathbf{C}_{\mathrm{Swan}}(X) \cong \mathbf{C}_{\mathrm{GS}}(\mathcal{O}_{\mathcal{B}})$$

where $\mathcal{B}$ is the acyclic basis of all afine opens.

7.2. **Presheaves of modules over presheaves of rings.** Let $\mathcal{B}$ be a poset and let $\mathcal{O}$ be a presheaf of k-algebras on $\mathcal{B}$. Let $\mathsf{Pr}(\mathcal{O})$ be the category of $\mathcal{O}$-modules.

Associated to $\mathcal{O}$ is a small k-linear category $\mathfrak{b}$ with $\mathrm{Ob}(\mathfrak{b}) = \mathcal{B}$ and

$$\mathfrak{b}(U, V) = \begin{cases} \mathcal{O}(U) & \text{if } U \subset V \\ 0 & \text{otherwise} \end{cases}$$

Proposition 7.2.1. *There is a quasi-isomorphism*

$$(7.8) \qquad\qquad \mathbf{C}_{\mathrm{ab}}(\mathsf{Pr}(\mathcal{O})) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{b}).$$

Proof. For $V \in \mathcal{B}$ let $P_V = \mathfrak{b}(-, V)$ be the extension by zero of $\mathcal{O} \mid V$. The $(P_V)_V$ form a system of small projective generators for $\mathsf{Pr}(\mathcal{O})$ and $\mathfrak{b} \longrightarrow \mathsf{Pr}(\mathcal{O}) : V \longmapsto P_V$ yields an equivalence of categories

$$\mathsf{Mod}(\mathfrak{b}) \cong \mathsf{Pr}(\mathcal{O}).$$

Hence the result is just a rephrazing of Corollary 6.9. $\qquad\qquad \square$

We now discuss the relation with the papers [13, 15] by Gerstenhaber-Schack. These authors work with relative Hochschild cohomology which makes it somewhat difficult to translate their results to our situation. So for simplicity we assume that k is a field. The Hochschild complex of $\mathcal{O}$ according to [13, 15] is

$$\mathbf{C}_{\mathrm{GS}}(\mathcal{O}) \stackrel{\mathrm{def}}{=} \mathrm{RHom}_{\mathcal{O}^{\mathrm{op}} \otimes \mathcal{O}}(\mathcal{O}, \mathcal{O})$$

Theorem 7.2.2. *There is a quasi-isomorphism*

$$(7.9) \qquad \mathbf{C}_{\mathrm{ab}}(\mathsf{Pr}(\mathcal{O})) \cong \mathbf{C}_{\mathrm{GS}}(\mathcal{O})$$

Proof. Let the $(P_V)_V$ be as in the proof of the previous proposition and let $\mathcal{O}!$ be the endomorphism algebra of the projective generator $P = \coprod_L P_L$ of $\mathsf{Pr}(\mathcal{O})$. The main result of [15] is the difficult "Special Cohomology Comparison Theorem":

$$\mathrm{RHom}_{\mathcal{O}^{\mathrm{op}} \otimes \mathcal{O}}(\mathcal{O}, \mathcal{O}) \cong \mathrm{RHom}_{\mathcal{O}!^{\mathrm{op}} \otimes \mathcal{O}!}(\mathcal{O}!, \mathcal{O}!)$$

By Proposition 5.5.1 and (2.5) we have

$$\mathrm{RHom}_{\mathcal{O}!^{\mathrm{op}} \otimes \mathcal{O}!}(\mathcal{O}!, \mathcal{O}!) \cong \mathbf{C}_{\mathrm{ab}}(\mathsf{Pr}(\mathcal{O}))$$

This finishes the proof. $\qquad\square$

7.3. Sheaves of modules over sheaves of rings.
Let $(X, \mathcal{O})$ be a ringed space. We prove (7.3).

Theorem 7.3.1. *Suppose $\mathcal{B}$ is a basis of X of* acyclic opens *and put $\mathfrak{b} = \mathfrak{u}(\mathcal{B})$. There is a quasi-isomorphism*

$$\mathbf{C}(X) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{b}).$$

Proof. Consider the composition

$$j : \mathfrak{b} \longrightarrow \mathsf{Pr}(\mathcal{O}) \longrightarrow \mathsf{Mod}(\mathcal{O}) : U \longmapsto i_{U!}\mathcal{O}_U$$

where $i_{U!}\mathcal{O}_U$ is the sheafification of P_U. Since $\mathcal{U}$ is a basis for the topology, j induces a localization (see for example [29]). For $U \subset V$, we have

$$\mathrm{Ext}^i_{\mathcal{O}}(i_{U!}\mathcal{O}_U, i_{V!}\mathcal{O}_V) = \mathrm{Ext}^i_{\mathcal{O}_U}(\mathcal{O}_U, (i_{V!}\mathcal{O}_V)|U) = H^i(U, \mathcal{O}_U)$$

and hence

    (1) $\mathfrak{b}(U, V) \longrightarrow \mathsf{Mod}(\mathcal{O})(i_{U!}\mathcal{O}_V, i_{V!}\mathcal{O}_V)$ is an isomorphism
    (2) $\mathrm{Ext}^i_{\mathcal{O}}(i_{U!}\mathcal{O}_V, i_{V!}\mathcal{O}_V) = 0$ for $i > 0$

So if we endow $\mathrm{Ob}(\mathfrak{b})$ with the censoring relation

$$(U, V) \in \mathcal{R} \iff U \subset V$$

we see that the result follows from Proposition 5.5.1. $\qquad\square$

Remark 7.3.2. The use of the censoring relation $\mathcal{R}$ is essential in the above proof as we have no control over $\mathrm{Ext}^i_{\mathcal{O}}(i_{U!}\mathcal{O}_U, i_{V!}\mathcal{O}_V)$ when $U \not\subset V$.

7.4. Constant sheaves.
In this section we prove (7.2). I.e. for a topological space X there is an isomorphism $H\mathbf{C}^*(X, \underline{k}) \cong H^*(X, \underline{k})$. The proof is basically a concatenation of some standard facts about cohomology of presheaves and sheaves.

If $\mathcal{C}$ is a small category and F is a presheaf of k-modules on $\mathcal{C}$ then the (presheaf!) cohomology $H^*(\mathcal{C}, F)$ of F is defined as the evaluation at F of the right derived functor $R^* \mathrm{proj}\lim$ of the inverse limit functor over $\mathcal{C}^{\mathrm{op}}$. It is well-known that $H^*(\mathcal{C}, F)$ can be computed with simplicial methods [25, 34, 35]. To be more precise put

$$F^n = \prod_{C_{i_0} \xrightarrow{f_{i_1}} \dots \xrightarrow{f_{i_n}} C_{i_n}} F(C_{i_0})$$

where, as the notation indicates, the product runs over all n-tuples of composable morphisms. Then $F^\bullet = (F^n)_n$ is a cosimplicial k-module and $H^*(\mathcal{C}, F)$ is the homology of the associated standard complex

$$0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \cdots$$

where the differentials are the usual alternating sign linear combinations of the boundary maps in $F^\bullet$.

Now let $k\mathcal{C}$ be the k-linear category with the same objects as $\mathcal{C}$ and Hom-sets given by

$$(k\mathcal{C})(C, D) = k^{\oplus \mathcal{C}(C,D)}$$

It is clear that $\mathcal{C} \mapsto k\mathcal{C}$ is the left adjoint to the forgetful functor from k-linear categories to arbitrary categories. The formula for F^n may be rewritten as

$$F^n = \prod \mathrm{Hom}_k((k\mathcal{C})(C_{i_n}, C_{i_{n+1}}) \otimes_k \cdots \otimes_k (k\mathcal{C})(C_{i_0}, C_{i_1}), F(C_{i_0}))$$

where now the product runs over $n + 1$ tuples of objects in $\mathcal{C}$.

To simplify even further assume that $\mathcal{C}$ is a poset. We make F into a $k\mathcal{C} - k\mathcal{C}$ bimodule in the following way

$$F(C, D) = F(C)$$

The dependence of $F(C, D)$ on D is as follows: if $f : D \longrightarrow D'$ is a map in $\mathcal{C}$ and $\alpha \in k$ then αf is the map from $F(C, D) = F(C)$ to $F(C, D') = F(C)$ given by multiplication by α. With this definition we may rewrite F^n once again as

$$F^n = \mathbf{C}^n(k\mathcal{C}, F)$$

Now let X be a topological space and put $\mathcal{C} = \mathsf{Open}(X)$, ordered by inclusion. In addition put $\mathfrak{c} = k\mathcal{C}$. Let $\underline{k}^p$ be the constant presheaf on X with values in k and let $\mathcal{O} = \underline{k}$ be its sheafification. We put $\mathsf{Mod}(X)$ for $\mathsf{Mod}(X, \mathcal{O}) = \mathsf{Mod}(X, \underline{k}^p)$, and similarly $\mathrm{Pr}(X) = \mathrm{Pr}(X, \underline{k}^p)$. Let $\epsilon : \mathsf{Mod}(X) \longrightarrow \mathrm{Pr}(X)$ be the inclusion functor.

Convention. *To avoid some confusing notations in this section, the sections of a (pre)sheaf $\mathcal{G}$ on an open U will always be denoted by $\Gamma(U, \mathcal{G})$ and not by $\mathcal{G}(U)$.*

If $\mathcal{F} \in \mathrm{Pr}(X)$ then we have

$$R \, \mathrm{proj} \lim_{\mathcal{C}^{\mathrm{op}}} (\mathcal{F}) = \mathrm{proj} \lim_{\mathcal{C}^{\mathrm{op}}} (\mathcal{F}) = \Gamma(X, \mathcal{F})$$

since $\mathcal{C}^{\mathrm{op}} = \mathsf{Open}(X)^{\mathrm{op}}$ has an initial object X. Applying this to an injective resolution $0 \longrightarrow \mathcal{G} \longrightarrow I^\cdot$ of an object $\mathcal{G}$ in $\mathsf{Mod}(X)$ we find

$$R\Gamma(X, \mathcal{G}) = \Gamma(X, I^\cdot) = \Gamma(X, \epsilon I^\cdot) = R \, \mathrm{proj} \lim(\epsilon I^\cdot)$$

and hence

$$H^*(X, \mathcal{G}) = H^*(\mathbf{C}(\mathfrak{c}, I^\cdot))$$

where we have suppressed the ϵ. We will construct an isomorphism

(7.10) $$H^*(\mathbf{C}(\mathfrak{c}, I^\cdot)) \cong H^*(\mathbf{C}_{\mathrm{ab}}(\mathsf{Mod}(X)))$$

for a specific choice of $I^\cdot$.

Since $\mathsf{Mod}(\mathfrak{c}) \cong \mathrm{Pr}(X)$, the functor

$$j : \mathfrak{c} \longrightarrow \mathsf{Mod}(X) : U \mapsto i_{U!}\mathcal{O}_U$$

defines a localization and hence the results from §5.1 apply. For $U \in \mathrm{Ob}(\mathfrak{c})$ we choose functorial injective resolutions $U \mapsto E(U)$ of $i_{U!}\mathcal{O}_U$ as in Remark 5.2.4.

For $U, V \in \mathcal{C}$ put

$$\bar{\bar{\mathfrak{c}}}(U, V) = \mathrm{Hom}_{\mathsf{Mod}(X)}(E(U), E(V))$$

According to Lemma 5.4.2 we have

$$\mathbf{C}_{\mathrm{ab}}(\mathsf{Mod}(X)) = \mathbf{C}(\mathfrak{c}, \bar{\bar{\mathfrak{c}}})$$

We prove the isomorphism (7.10) for $I^{\cdot} = E(X)$. To this end it is sufficient by Proposition 4.3.1 to construct a quasi-isomorphism between the complexes of $\mathfrak{c}$-bimodules $E(X)_0$ and $\bar{\bar{\mathfrak{c}}}_0$. I.e. for $U \subset V$ we must construct quasi-isomorphisms between $E(X)(U, V)$ and $\bar{\bar{\mathfrak{c}}}(U, V)$ which are natural in U, V. Recall that in the current setting

$$E(X)(U, V) = \Gamma(U, E(X))$$

We have quasi-isomorphisms

$$\bar{\bar{\mathfrak{c}}}(U, V) = \mathrm{Hom}_X(E(U), E(V)) \xrightarrow{\cong} \mathrm{Hom}_X(i_{U!}\mathcal{O}_U, E(V)) \xrightarrow{\cong}$$

$$\Gamma(U, E(V)) \xrightarrow{\cong} \Gamma(U, E(X))$$

The last arrow is obtained from the map $E(V) \longrightarrow E(X)$ which comes from the map $V \longrightarrow X$ by functoriality. To see that it is a quasi-isomorphism note that $i_{V!}(\mathcal{O}_V) \mid U \cong \mathcal{O} \mid U$ implies that $E(V) \mid U \longrightarrow E(X) \mid U$ is a quasi-isomorphism. Since $E(V) \mid U$ and $E(X) \mid U$ consist of injectives we obtain indeed a quasi-isomorphism between $\Gamma(U, E(V))$ and $\Gamma(U, E(X))$.

7.5. Sheaves of modules over a quasi-compact, separated scheme.

In this section we prove (7.5). Let X be a quasi-compact separated scheme and let $X = \bigcup_{i=1}^{n} A_i$ be a finite affine covering of X. For $J \subset I = \{1, 2, \ldots, n\}$, put $A_J = \bigcap_{i \in J} A_i$. Each A_J is affine since X is separated. Put $\mathcal{A} = \{A_J \mid \varnothing \neq J \subset I\}$.

Theorem 7.5.1. *There is a quasi-isomorphism*

$$\mathbf{C}(X) \cong \mathbf{C}_{\mathrm{sh}}(\mathfrak{u}(\mathcal{A})).$$

Proof. Let $\mathcal{X}$ be the collection of all open subsets in X and fix once and for all a k-cofibrant resolution $\overline{\mathfrak{u}(\mathcal{X})} \longrightarrow \mathfrak{u}(\mathcal{X})$ with $\overline{\mathfrak{u}(\mathcal{X})}(U, V) = 0$ if U is not in V (recall that we can achieve this by replacing an arbitrary k-cofibrant resolution $\overline{\mathfrak{u}(\mathcal{X})}$ by $\overline{\mathfrak{u}(\mathcal{X})}_0$).

All resolutions will be chosen to be restrictions of $\overline{\mathfrak{u}(\mathcal{X})} \longrightarrow \mathfrak{u}(\mathcal{X})$. If $\mathcal{C}$, $\mathcal{D}$ are collections of opens and $\mathfrak{c}$ is the k-cofibrant category corresponding to $\mathcal{C}$, $\mathfrak{c}_{\mathcal{D}}$ contains the opens $U \in \mathcal{C}$ with $U \subset D$ for some $D \in \mathcal{D}$, and $\mathfrak{c}_D = \mathfrak{c}_{\{D\}}$. We write $\mathcal{C} \leq \mathcal{D}$ (or $\mathfrak{c} \leq \mathcal{D}$) if for every $C \in \mathcal{C}$, there is a $D \in \mathcal{D}$ with $C \subset D$.

Let $\mathcal{B}$ be a basis of affine opens and $\mathcal{A}'$ a collection of affine opens with $\mathcal{A} \subset \mathcal{A}' \subset \mathcal{B}$. Let $\mathfrak{a}$, $\mathfrak{a}'$ and $\mathfrak{b}$ denote the corresponding k-cofibrant categories. We will prove that the induced map $\mathbf{C}(\mathfrak{b}) \longrightarrow \mathbf{C}(\mathfrak{a}'_{\mathcal{A}})$ is a quasi-isomorphism. Taking $\mathcal{A}' = \mathcal{A}$, the result then follows from Theorem 7.3.1. The proof goes by induction on the number n of affine opens in the covering used to produce $\mathcal{A}$.

For $n = 1$, the statement follows from the Lemma 7.5.2 below. For arbitrary n, put $\mathcal{A}_1 = \{A_i\}_{i=2}^{n}$ and $\mathcal{A}_2 = \{A_1 \cap A_i\}_{i=2}^{n}$. For any $\mathfrak{c} \leq \mathcal{A}$, we get an exact sequence of double complexes (cfr. §2.4)

$$0 \longrightarrow \mathbf{D}(\mathfrak{c}) \longrightarrow \mathbf{D}(\mathfrak{c}_{A_1}) \oplus \mathbf{D}(\mathfrak{c}_{A_1}) \longrightarrow \mathbf{D}(\mathfrak{c}_{\mathcal{A}_2}) \longrightarrow 0$$

since $\mathfrak{c}(C_{p-1}, C_p) \otimes \cdots \otimes \mathfrak{c}(C_0, C_1)$ can be different from zero only if $C_0 \subset C_1 \subset \cdots \subset C_{p-1} \subset C_p$. Applying this with $\mathfrak{c} = \mathfrak{b}, \mathfrak{a}'$, we obtain a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{C}(\mathfrak{b}_{\mathcal{A}}) & \longrightarrow & \mathbf{C}(\mathfrak{b}_{A_1}) \oplus \mathbf{C}(\mathfrak{b}_{A_1}) & \longrightarrow & \mathbf{C}(\mathfrak{b}_{\mathcal{A}_2}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{C}(\mathfrak{a}'_{\mathcal{A}}) & \longrightarrow & \mathbf{C}(\mathfrak{a}'_{A_1}) \oplus \mathbf{C}(\mathfrak{a}'_{A_1}) & \longrightarrow & \mathbf{C}(\mathfrak{a}'_{\mathcal{A}_2}) & \longrightarrow & 0
\end{array}
$$

It suffices to prove that the arrow to the left is a quasi-isomorphism. This follows from the induction hypothesis applied to the three maps to the right. $\qquad \square$

Lemma 7.5.2. *Suppose X is affine and let $\mathcal{C}$ be a collection of affine subsets with $X \in \mathcal{C}$. Then $\mathfrak{u}(\{X\}) \longrightarrow \mathfrak{u}(\mathcal{C})$ induces a quasi-isomorphism $\mathbf{C}(\overline{\mathfrak{u}(\mathcal{C})}) \longrightarrow \mathbf{C}(\overline{\mathfrak{u}(\{X\})})$ for restrictions of $\overline{\mathfrak{u}(\mathcal{X})} \longrightarrow \mathfrak{u}(\mathcal{X})$.*

Proof. Put $\mathfrak{c} = \overline{\mathfrak{u}(\mathcal{C})}$ and $\mathfrak{y} = \overline{\mathfrak{u}(\{X\})}$. We endow $\mathfrak{c}$ with a censoring relation $(U, V) \in \mathcal{R} \iff U \subset V$. and we use Proposition 4.3.4 for the inclusion $\mathfrak{y} \subset \mathfrak{c}$.

For $U \subset V \in \mathcal{C}$, we compute

$$
\begin{aligned}
\mathrm{RHom}_{\mathfrak{y}^{\mathrm{op}}}(\mathfrak{c}(V, -), \mathfrak{c}(U, -)) &\cong \mathrm{RHom}_{\mathcal{O}(X)^{\mathrm{op}}}(\mathcal{O}(V), \mathcal{O}(U)) \\
&\cong \mathrm{RHom}_{\mathsf{Qch}(\mathcal{O}_X)}(i_{V,*}\mathcal{O}_V, i_{U,*}\mathcal{O}_U) \\
&\cong \mathrm{RHom}_{\mathsf{Mod}(\mathcal{O}_X)}(i_{V,*}\mathcal{O}_V, i_{U,*}\mathcal{O}_U) \\
&\cong \mathrm{RHom}_{\mathsf{Mod}(\mathcal{O}_U)}((i_{V,*}\mathcal{O}_V)|_U, \mathcal{O}_U) \\
&\cong \mathrm{RHom}_{\mathsf{Mod}(\mathcal{O}_U)}(\mathcal{O}_U, \mathcal{O}_U) \\
&\cong \mathcal{O}(U) \\
&\cong \mathfrak{c}(U, V)
\end{aligned}
$$

where the first line follows from the fact that $\Gamma(X, -)$ defines an equivalence between $\mathsf{Mod}(\mathcal{O}(X))$ and $\mathsf{Qch}(X)$ and the third line follows from the separatedness of affine schemes [40, Appendix B]. $\qquad \square$

7.6. **Computing** RHom**'s using a covering.** Let X be a quasi-compact quasi-separated scheme and let $X = \cup_{i=1}^n A_i$ be as in the previous section. We use the same associated notations. Let $\mathsf{Qch}(X)$ be the category of quasi-coherent sheaves on X. In this section we prove that the functor $\mathsf{Qch}(X) \longrightarrow \mathsf{Pr}(\mathcal{O}_{\mathcal{A}})$ preserves RHom. The actual reason for this is that one may show that the simplicial scheme $S_\bullet$ defined by $S_n = \coprod_{i_1 \leq \cdots \leq i_n} A_{i_1} \cap \cdots \cap A_{i_n}$ satisfies "effective cohomological descent" [4, Exposé V$^{\mathrm{bis}}$] for the obvious map $\epsilon : S_\bullet \longrightarrow X$, even though it is not quite a hypercovering in the sense of [4, Exposé V7] (to obtain a hypercovering we need to take all sequences $(i_1, \cdots, i_n)$ and not just the ordered ones). For the convenience of the reader we will give a direct proof of the preservation of RHom in our special case.

Before giving the proof let us give a quick sketch. Let $\epsilon^* : \mathsf{Qch}(X) \longrightarrow \mathsf{Pr}(\mathcal{O}_{\mathcal{A}})$ be the exact inclusion functor. It is easy to see that ϵ^* has a right adjoint ϵ_*, which is some kind of global section functor, satifying $\epsilon_* \epsilon^* = \mathrm{id}$.

We then prove that ϵ_* sends injective objects in $\mathsf{Qch}(X)$ to acyclic objects for ϵ^*. Thus we obtain $R\epsilon_* \circ \epsilon^* = \mathrm{id}$ and hence ϵ^* is fully faithful for RHom.

For convenience, let $\tilde{\Delta}$ be the poset $\{I \mid I \subset \{1, \ldots, n\}\}$ ordered by reversed inclusion and let Δ be its subposet of all $J \neq \varnothing$. For $I \in \tilde{\Delta}$ put $A_I = \bigcap_{i \in A} A_i$ (with

$A_\varnothing = X$). We have maps $\Delta \longrightarrow \tilde{\Delta} \longrightarrow \mathsf{Open}(X) : I \longmapsto A_I$ (with $A_\varnothing = X$) which allow us to consider the restrictions $\mathcal{O}_{\tilde{\Delta}}$ and $\mathcal{O}_\Delta$ of $\mathcal{O}$. We will think of $\mathsf{Pr}(\mathcal{O}_\Delta)$ as (equivalent to) the "category of (presheaf) objects in the stack of abelian categories $\mathsf{Qch} : \Delta \longrightarrow \mathsf{Cat} : I \longmapsto \mathsf{Qch}(A_I)$".

In order to abstract the reasoning we will formulate our results in the following somewhat more general setting. $\tilde{\mathcal{S}}$ will be a stack of Grothendieck categories on $\tilde{\Delta}$ with exact restriction functors possessing a fully faithful right adjoint, and $\mathcal{S}$ will be its restriction to Δ.

For $I \supset J$, we write

$$i_{IJ}^* : \mathcal{S}(J) \longrightarrow \mathcal{S}(I)$$

for the exact restriction functor and

$$i_{IJ*} : \mathcal{S}(I) \longrightarrow \mathcal{S}(J).$$

for its fully faithful right adjoint. We will put $i_{I\varnothing}^* = i_I^*$ and $i_{I\varnothing*} = i_{I*}$. Besides the above mentioned properties we will also use the following properties

(C1) (Base Change) $i_I^* i_{J*} = i_{I\cup J,I*} i_{I\cup J,J}^*$.

(C2) If $E \in \tilde{\mathcal{S}}(\varnothing)$ is injective, then for every $K \in \Delta$, $i_K^* E$ is acyclic for i_{K*}.

It is not clear to us if (C1) does not follow from the other properties. (C2) will be only used in the proof of Theorem 7.6.6 which is the main result of this section. The additional conditions are clearly satisfied for the stacks $U \mapsto \mathsf{Mod}(U)$ on a finite cover of a ringed space and for $U \mapsto \mathsf{Qch}(U)$ on a finite affine cover of a separated scheme.

A *(presheaf)object in* $\mathcal{S}$ consists of objects $(M_I)_I$ with $M_I \in \mathcal{S}(I)$ and maps $(\phi_{IJ})_{I \supset J}$ with $\phi_{IJ} : i_{IJ}^* M_J \longrightarrow M_I$ for $I \supset J$ satisfying the obvious compatibilities. Presheaf objects in $\mathcal{S}$ and the obvious compatible morphisms constitute a Grothendieck category $\mathsf{Pr}(\mathcal{S})$.

Proposition 7.6.1. *The exact functor*

$$j_I^* : \mathsf{Pr}(\mathcal{S}) \longrightarrow \mathcal{S}(K) : (M_K)_K \longmapsto M_I$$

has a right adjoint j_{I*} *with*

$$(j_{I*}M)_J = \begin{cases} i_{IJ*}M & \text{if } I \supset J \\ 0 & \text{otherwise} \end{cases}$$

Proposition 7.6.2. *The exact functor*

$$x : \tilde{\mathcal{S}}(\varnothing) \longrightarrow \mathsf{Pr}(\mathcal{S}) : M \longmapsto (i_I^* M)_I$$

has a right adjoint

$$y : \mathsf{Pr}(\mathcal{S}) \longrightarrow \tilde{\mathcal{S}}(\varnothing) : (M_K)_K \longmapsto \mathop{\mathrm{proj\,lim}}_{K}(i_{K*}M_K).$$

Proof. For $M \in \tilde{\mathcal{S}}(\varnothing)$, $(N_K)_K \in \mathsf{Pr}(\mathcal{S})$, we have

$$\mathrm{Hom}_{\tilde{\mathcal{S}}(\varnothing)}(M, y(N_K)_K) = \mathrm{Hom}_{\tilde{\mathcal{S}}(\varnothing)}(M, \mathop{\mathrm{proj\,lim}}_{K}(i_{K*}N_K))$$

$$= \mathop{\mathrm{proj\,lim}}_{K} \mathrm{Hom}_{\tilde{\mathcal{S}}(\varnothing)}(M, i_{K*}N_K)$$

$$= \mathop{\mathrm{proj\,lim}}_{K} \mathrm{Hom}_{\tilde{\mathcal{S}}(\varnothing)}(i_K^* M, N_K)$$

$$= \mathrm{Hom}_{\mathsf{Pr}(\mathcal{S})}(\epsilon^* M, (N_K)_K) \quad \square$$

For every $K \in \Delta$, we obtain two commutative diagrams:

$$
\begin{array}{ccc}
& \tilde{\mathcal{S}}(\varnothing) & \\
{}^{x}\swarrow & \downarrow{}^{i_K^*} & \\
\mathsf{Pr}(\mathcal{S}) \xrightarrow[j_K^*]{} & \mathcal{S}(K) &
\end{array}
\qquad
\begin{array}{ccc}
& \tilde{\mathcal{S}}(\varnothing) & \\
{}^{y}\nearrow & \uparrow{}^{i_{K*}} & \\
\mathsf{Pr}(\mathcal{S}) \xleftarrow[j_{K*}]{} & \mathcal{S}(K) &
\end{array}
$$

The arrows in the left triangle are exact left adjoints to the corresponding arrows in the right triangle, which consequently preserve injectives.

Let $\mathfrak{e}$ denote the full subcategory of $\mathsf{Pr}(\mathcal{S})$ spanned by the objects $j_{K*}E$ for E injective in $\mathcal{S}(K)$ and $K \in \Delta$. Let $\mathsf{add}(\mathfrak{e})$ be spanned by all finite sums of objects in $\mathfrak{e}$.

Proposition 7.6.3. $\mathsf{Pr}(\mathcal{S})$ *has enough injectives in* $\mathsf{add}(\mathfrak{e})$.

Proof. For $M = (M_K)_K$ in $\mathsf{Pr}(\mathcal{S})$, we can choose monomorphisms $M_K \longrightarrow E_K$ in $\mathcal{S}(K)$ for each K. The corresponding maps $M \longrightarrow j_{K*}E_K$ yield a map

$$
M \longrightarrow \bigoplus_K j_{K*}E_K,
$$

which is a monomorphism since every image under j_K^* is. $\qquad\square$

We will now describe $R\epsilon_*$ for certain $N \in \mathsf{Pr}(\mathcal{S})$. To $N = (N_K)_K$ in $\mathsf{Pr}(\mathcal{S})$, we associate the following complex $S(N)$ in $C(\tilde{\mathcal{S}}(\varnothing))$:

$$
\prod_p i_{p*}N_p \longrightarrow \prod_{p<q} i_{pq*}N_{pq} \longrightarrow \ldots \longrightarrow i_{1\ldots n*}N_{1\ldots n} \longrightarrow 0
$$

with $\prod_p i_{p*}N_p$ in degree zero and with the usual alternating sign differentials $d_N^0, d_N^1, \ldots$.

Proposition 7.6.4. *For* $N = (N_K)_K$ *in* $\mathsf{Pr}(\mathcal{S})$, *we have*

$$
\epsilon_* N = H^0(S(N))
$$

If for every K, N_K *is acyclic for* i_{K*}, *we have in* $D(\tilde{\mathcal{S}}(\varnothing))$

$$
R\epsilon_* N = S(N).
$$

Proof. First note that it is clear from the shape of Δ that if $(F_K)_{K\in\Delta}$ is a functor $\Delta \longrightarrow \tilde{\mathcal{S}}(\varnothing)$, then $\mathrm{proj}\lim_K F_K = \mathrm{proj}\lim_{|K|\in\{1,2\}} F_K$. Hence the first statement follows from Proposition 7.6.2. For the second statement, first consider $N = j_{K*}E \in \mathfrak{e}$. By Proposition 7.6.1, the complex $S(N)$ is given by

$$
\prod_{p\in K} i_{K*}E \longrightarrow \prod_{p<q\in K} i_{K*}E \longrightarrow \ldots \longrightarrow i_{K*}E \longrightarrow 0.
$$

This complex is acyclic, except in degree zero where its homology is $i_{K*}E = \epsilon_* N$. It follows that for all $E \in \mathsf{add}(\mathfrak{e})$, $S(E) = \epsilon_* E = R\epsilon_* E$. Now consider N with every N_K acyclic for i_{K*}. Take a resolution $N \longrightarrow E^{\cdot}$ of N in $\mathsf{add}(\mathfrak{e})$. Consider $S(E^{\cdot})$ as a first quadrant double complex with the complexes $S(E^i)$ vertical. Looking at columns first, we find a quasi-isomorphism $\mathrm{Tot}(S(E^{\cdot})) \cong \epsilon_* E^{\cdot} \cong R\epsilon_* N$. Since for $E \in \mathsf{add}(\mathfrak{e})$, the E_K are obviously acyclic for i_{K*}, it follows from our assumption on N that $i_{K*}(N_K \longrightarrow E_K^{\cdot})$ is exact. Hence looking at rows, we find a quasi-isomorphism $\mathrm{Tot}(S(E^{\cdot})) \cong S(N)$, which finishes the proof. $\qquad\square$

Proposition 7.6.5. *For $M \in \tilde{\mathcal{S}}(\varnothing)$, we have*

(1) $\epsilon_* \epsilon^* M = M$;

(2) $S(\epsilon^* M) = M$ *in* $D(\tilde{\mathcal{S}}(\varnothing))$;

Proof. To prove the two statements, it suffices that the complex

$$0 \longrightarrow M \longrightarrow \prod_p i_{p*} i_p^* M \longrightarrow \prod_{p<q} i_{pq*} i_{pq}^* M \longrightarrow \ldots \longrightarrow i_{1\ldots n*} i_{1\ldots n}^* M \longrightarrow 0$$

is acyclic. This follows from the fact that for every $r \in \{1, \ldots n\}$, the image of the complex under i_r^* has a contracting chain homotopy. $\qquad\square$

Theorem 7.6.6. *The functor $\epsilon^* : \tilde{\mathcal{S}}(\varnothing) \longrightarrow \mathsf{Pr}(\mathcal{S})$ induces a fully faithful functor $\epsilon^* : D^+(\tilde{\mathcal{S}}(\varnothing)) \longrightarrow D^+(\mathsf{Pr}(\mathcal{S}))$*

Proof. It is sufficient to prove that $R\epsilon_* \circ \epsilon^*$ is the identity on $D^+(\tilde{\mathcal{S}}(\varnothing))$. To this end it is sufficient to prove that $\epsilon_* \epsilon^*$ is the identity and that ϵ^* sends injectives to acyclic objects for ϵ_*. The first part is Proposition 7.6.5(1). The second part follows from Proposition 7.6.5(1), the extra condition (C2) on $\mathcal{S}$ and Proposition 7.6.4. $\qquad\square$

7.7. Quasi-coherent sheaves over a quasi-compact, separated scheme.
We keep the notations of the previous section. The following theorem is still true in the general setting exhibited there.

Theorem 7.7.1. *There is a quasi-isomorphism*

$$\mathbf{C}_{\mathrm{ab}}(\tilde{\mathcal{S}}(\varnothing)) \cong \mathbf{C}_{\mathrm{ab}}(\mathsf{Pr}(\mathcal{S})).$$

Proof. Let $\mathfrak{e}$ be as before and let $\mathfrak{i}(\varnothing)$ be the category of injectives in $\tilde{\mathcal{S}}(\varnothing)$. Consider the $\mathfrak{e}-\mathfrak{i}(\varnothing)$-bimodule X with

$$X(I, j_{K*}E) = \mathrm{Hom}_{\mathsf{Pr}(\mathcal{S})}(\epsilon^* I, j_{K*}E) = \mathrm{Hom}_{\tilde{\mathcal{S}}(\varnothing)}(I, i_{K,*}E).$$

For $I, J \in \mathfrak{i}(\varnothing)$, we compute:

$$\begin{aligned}
\mathrm{RHom}_{\mathfrak{e}^{\mathrm{op}}}(X(J, -), X(I, -)) &= \mathrm{RHom}_{\mathfrak{e}^{\mathrm{op}}}(\mathrm{Hom}_{\mathsf{Pr}(\mathcal{S})}(\epsilon^* J, -), \mathrm{Hom}_{\mathsf{Pr}(\mathcal{S})}(\epsilon^* I, -)) \\
&= \mathrm{RHom}_{\mathsf{Pr}(\mathcal{S})}(\epsilon^* I, \epsilon^* J) \\
&= \mathrm{RHom}_{\tilde{\mathcal{S}}(\varnothing)}(I, J) \\
&= \mathrm{Hom}_{\tilde{\mathcal{S}}(\varnothing)}(I, J)
\end{aligned}$$

where the second step follows from Lemma 5.3.3 and the third step is Theorem 7.6.6. For $j_{K*}E$ and $j_{L*}F$ in $\mathfrak{e}$, first note that

$$\mathrm{Hom}_{\mathsf{Pr}(\mathcal{S})}(j_{K*}E, j_{L*}F) = \mathrm{Hom}_{\mathcal{S}(L)}(j_L^* j_{K*}E, F),$$

which equals zero unless $L \subset K$, and equals

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{S}(L)}(i_{KL*}E, F) &= \mathrm{Hom}_{\tilde{\mathcal{S}}(\varnothing)}(i_{K*}E, i_{L*}F) \\
&= \mathrm{Hom}_{\mathfrak{e}}(i_{K*}E, i_{L*}F)
\end{aligned}$$

if $L \subset K$ since i_{L*} is fully faithful. For $L \subset K$, we now compute

$$\begin{aligned}
\mathrm{RHom}_{\mathfrak{i}(\varnothing)}(\mathrm{Hom}_{\tilde{\mathcal{S}}(\varnothing)}(-, i_{K*}E), \mathrm{Hom}_{\tilde{\mathcal{S}}(\varnothing)}(-, i_{L*}F)) &= \mathrm{Hom}_{\mathfrak{i}(\varnothing)}(i_{K*}E, i_{L*}F) \\
&= \mathrm{Hom}_{\mathfrak{e}}(j_{K*}E, j_{L*}F) \\
&= \mathrm{Hom}_{\mathfrak{e}}(i_{K*}E, i_{L*}F)
\end{aligned}$$

We endow $\mathfrak{e}$ with the censoring relation $(j_{K*}E, j_{L*}F) \in \mathcal{R} \iff j_{K*}E \neq 0 \neq j_{L*}F$ and $L \subset K$. By Proposition 4.3.3 and Corollary 6.7 we obtain the desired quasi-isomorphism. $\qquad\square$

Specializing to quasi-coherent sheaves on a quasi-compact separated scheme we get

Corollary 7.7.2. *There is a quasi-isomorphism*

$$\mathbf{C}_{\mathrm{ab}}(\mathsf{Qch}(X)) \cong \mathbf{C}_{\mathrm{ab}}(\mathsf{Pr}(\mathcal{O}_{\mathcal{A}})).$$

and in the case of a noetherian separated scheme we get

Corollary 7.7.3. *There is a quasi-isomorphism*

$$\mathbf{C}_{\mathrm{ab}}(\mathsf{coh}(X)) \cong \mathbf{C}_{\mathrm{ab}}(\mathsf{Pr}(\mathcal{O}_{\mathcal{A}})).$$

using Corollary 6.8, since $\mathsf{Qch}(X) = \mathsf{Indcoh}(X)$.

Remark 7.7.4. By combining the above results we obtain an isomorphism in $\mathrm{Ho}(B_\infty)$

$$(7.11) \qquad\qquad \mathbf{C}_{\mathrm{ab}}(\mathsf{Qch}(X)) \cong \mathbf{C}_{\mathrm{ab}}(\mathsf{Mod}(X))$$

for a quasi-compact separated scheme X, but our proof of this fact is far from straightforward and goes through the auxilliary categories $\mathsf{Pr}(\mathcal{O}_{\mathcal{A}})$ and $\mathsf{Pr}(\mathcal{O}_{\mathcal{B}})$. It would be interesting to see if a more direct proof could be obtained.

7.8. **Relation to Swan's definition.** In [39], Richard G. Swan defined the Hochschild cohomology of a separated scheme X to be

$$\mathrm{Ext}^i_{X \times X}(\mathcal{O}_D, \mathcal{O}_D)$$

where $\mathcal{O}_D = \delta_* \mathcal{O}_X$ for the diagonal map $\delta : X \longrightarrow X \times X$. Put $\mathbf{C}_{\mathrm{Swan}}(X) = \mathrm{RHom}_{X \times X}(\mathcal{O}_D, \mathcal{O}_D)$.

We prove that Swan's definition coincides with ours. As was already mentioned this in the finite type case could be deduced from [39, §3] and the above results.

Theorem 7.8.1. *Let X be a quasi-compact, separated scheme over a field k with an affine covering $\mathcal{A}$ given by $X = \cup_{i=1}^n A_i$. There is a quasi-isomorphism*

$$\mathbf{C}_{\mathrm{Swan}}(X) \cong \mathbf{C}_{\mathrm{GS}}(\mathcal{O}_{\mathcal{A}}).$$

Proof. Consider the factorization of $\delta : X \longrightarrow X \times X$ over $\delta' : X \longrightarrow X'$ where $X' = \cup_{i=1}^n A_i \times A_i$ is the open subscheme of $X \times X$ with $\mathcal{O}_{X'}(U) = \mathcal{O}_{X \times X}(U)$ and in particular $\mathcal{O}_{X'}(A_i \times A_i) = \mathcal{O}_X(A_i) \otimes \mathcal{O}_X(A_i)$. Put $\mathcal{O}_{D'} = \delta'_* \mathcal{O}_X$. In particular, $\mathcal{O}_{D'}(A_i \times A_i) = \mathcal{O}_X(A_i)$. If we identify the collection $\mathcal{A}'$ associated to the covering $\cup_{i=1}^n A_i \times A_i$ of X' with $\mathcal{A}$, we have $\mathcal{O}_{X'\mathcal{A}'} = \mathcal{O}_{X\mathcal{A}} \otimes \mathcal{O}_{X\mathcal{A}}$ and $\mathcal{O}_{D'\mathcal{A}'} = \mathcal{O}_{X\mathcal{A}}$. Since X is separated, we have $\sup(\mathcal{O}_D) \subset \delta(X) \subset X'$, hence we may compute

$$\mathrm{RHom}_{X \times X}(\mathcal{O}_D, \mathcal{O}_D) = \mathrm{RHom}_{X'}(\mathcal{O}_{D'}, \mathcal{O}_{D'})$$
$$= \mathrm{RHom}_{\mathsf{Qch}(X')}(\mathcal{O}_{D'}, \mathcal{O}_{D'})$$
$$= \mathrm{RHom}_{\mathsf{Pr}(\mathcal{O}_{\mathcal{A}} \otimes \mathcal{O}_{\mathcal{A}})}(\mathcal{O}_{\mathcal{A}}, \mathcal{O}_{\mathcal{A}})$$

where we have used that X is quasi-compact, separated in the second step and we have used Theorem 7.6.6 in the last step. $\qquad\square$

Corollary 7.8.2. *We have*

$$\mathbf{C}(X) \cong \mathbf{C}_{\mathrm{Swan}}(X)$$

Proof. We have

$$\mathbf{C}_{\mathrm{Swan}}(X) \cong \mathbf{C}_{GS}(\mathcal{O}_{\mathcal{A}}) \cong \mathbf{C}(\mathsf{Pr}(\mathcal{O}_{\mathfrak{a}})) \cong \mathbf{C}(\mathfrak{u}(\mathfrak{a})) \cong \mathbf{C}(X)$$

The first isomorphism is Theorem 7.8.1, the second isomorphism is (7.9), the third isomorphism is (7.8) and the fourth isomorphism is Theorem 7.5.1.   □

7.9. The Mayer-Vietoris sequence. Let $(X, \mathcal{O})$ be a ringed space and let $X = U \cup V$. In this section we prove (7.1). If $W_1 \subset W_2$ are open embeddings of ringed spaces then the pushforward functor $i_{W_2, W_1 *}$ is fully faithful and preserves injectives. Hence it induces a restriction map $\mathbf{C}(W_2) \longrightarrow \mathbf{C}(W_1)$ (see Remark 4.2.3). It is clear that the restriction map is compatible with compositions. So $\mathbf{C}(-)$ defines a contravariant functor on open embeddings of ringed spaces.

Theorem 7.9.1. *There is an exact triangle of complexes*

(7.12) $$\mathbf{C}(X) \longrightarrow \mathbf{C}(U) \oplus \mathbf{C}(V) \longrightarrow \mathbf{C}(U \cap V) \longrightarrow$$

where the maps are the restriction maps defined above (in particular they are B_∞-maps).

Taking homology in (7.12) yields the Mayer-Vietoris sequence.

Proof. We use notations as in §7.6. Put $U_1 = U, U_2 = V$ and $\Delta = \{\{1\}, \{2\}, \{1, 2\}\}$. Let $\mathcal{S}$ be the stack of abelian categories $\mathsf{Mod}(-)$ associated to the covering $X = U_1 \cup U_2$. Let $\mathfrak{e}$ be the full subcategory of $\mathsf{Pr}(\mathcal{S})$ consisting of $j_{K*}E$ for E injective in $\mathcal{S}(K)$ and $K \in \Delta$.

Let $\mathfrak{e}_I = \mathsf{InjMod}(U_I)$ for $I \subset \{1, 2\}$. As mentioned above the functors $i_{I, \varnothing *} : \mathfrak{e}_I \longrightarrow \mathfrak{e}_\varnothing$ are fully faithful. Choose a k-cofibrant resolution $\bar{\mathfrak{e}}_\varnothing \longrightarrow \mathfrak{e}_\varnothing$ and let $\bar{\mathfrak{e}}_I \longrightarrow \mathfrak{e}_I$ be the restrictions of this resolution. Let $\bar{\mathfrak{e}}$ be the category with $\mathrm{Ob}(\bar{\mathfrak{e}}) = \coprod_{J \in \Delta} \mathrm{Ob}(\mathfrak{e}_J)$ and Hom-sets between $E \in \mathfrak{e}_I$ and $F \in \mathfrak{e}_J$ given by

$$\bar{\mathfrak{e}}(E, F) = \begin{cases} \bar{\mathfrak{e}}_\varnothing(E, F) & \text{if } J \subset I \\ 0 & \text{otherwise} \end{cases}$$

Then $\bar{\mathfrak{e}}$ is a k-cofibrant resolution of $\mathfrak{e}$.

Let $\bar{\mathfrak{e}}^I$ be the full-subcategory of $\bar{\mathfrak{e}}$ consisting of objects in $\bar{\mathfrak{e}}_J$ with $I \subset J$. We have $\bar{\mathfrak{e}}^{\{1,2\}} = \bar{\mathfrak{e}}_{\{1,2\}}$ and $\bar{\mathfrak{e}}^{\{j\}}$ is the arrow category $(\mathfrak{e}_{\{1,2\}} \longrightarrow \mathfrak{e}_{\{j\}})$ where the arrow is the inclusion.

Then by using restriction maps we obtain a commutative diagram of complexes

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{C}(\bar{\mathfrak{e}}) & \longrightarrow & \mathbf{C}(\bar{\mathfrak{e}}^{\{1\}}) \oplus \mathbf{C}(\bar{\mathfrak{e}}^{\{2\}}) & \longrightarrow & \mathbf{C}(\bar{\mathfrak{e}}^{\{1,2\}}) & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & \mathbf{C}(\bar{\mathfrak{e}}_{\{1\}}) \oplus \mathbf{C}(\bar{\mathfrak{e}}_{\{2\}}) & \longrightarrow & \mathbf{C}(\bar{\mathfrak{e}}_{\{1,2\}}) & \longrightarrow & 0
\end{array}
$$

The vertical maps are isomorphisms by Remark 4.1.5 and the top row is an exact sequence of complexes. Thus we obtain a triangle

$$\mathbf{C}(\bar{\mathfrak{e}}) \longrightarrow \mathbf{C}(\bar{\mathfrak{e}}_{\{1\}}) \oplus \mathbf{C}(\bar{\mathfrak{e}}_{\{2\}}) \longrightarrow \mathbf{C}(\bar{\mathfrak{e}}_{\{1,2\}}) \longrightarrow$$

By Theorem 7.7.1, Proposition 7.6.3 and Corollary 6.7 we have $\mathbf{C}(X) \cong \mathbf{C}(\bar{\mathfrak{e}}_\varnothing) \cong \mathbf{C}(\bar{\mathfrak{e}})$. This means that we are almost done, except for the fact that we still need to show that the composition

$$\mathbf{C}(\bar{\mathfrak{e}}_\varnothing) \xrightarrow{\cong} \mathbf{C}(\bar{\mathfrak{e}}) \longrightarrow \mathbf{C}(\bar{\mathfrak{e}}_{\{j\}})$$

for $j = 1, 2$ is the restriction map. To this end we recall that the isomorphism $\mathbf{C}(\bar{\mathfrak{e}}_\varnothing) \xrightarrow{\cong} \mathbf{C}(\bar{\mathfrak{e}})$ was constructed in the proof of Theorem 7.7.1 using the $\bar{\mathfrak{e}} - \bar{\mathfrak{e}}_\varnothing$ bimodule X where

$$X(E, F) = \mathrm{Hom}_{\mathsf{Pr}(\mathcal{S})}(\epsilon^* E, F) \qquad \text{for } E \in \bar{\mathfrak{e}}_\varnothing,\ F \in \bar{\mathfrak{e}}$$

We have a commutative diagram of inclusions:

$$
\begin{array}{ccccc}
\bar{\mathfrak{e}}_\varnothing & \xrightarrow{\cong} & (\bar{\mathfrak{e}}_\varnothing \xrightarrow{X} \bar{\mathfrak{e}}) & \xleftarrow{\cong} & \bar{\mathfrak{e}} \\
\| & & \uparrow & & \uparrow \\
\bar{\mathfrak{e}}_\varnothing & \xrightarrow[i_{\bar{\mathfrak{e}}_\varnothing}]{} & (\bar{\mathfrak{e}}_\varnothing \xrightarrow{X_j} \bar{\mathfrak{e}}_{\{j\}}) & \xleftarrow{} & \bar{\mathfrak{e}}_{\{j\}}
\end{array}
$$

where "$\cong$" means inducing an isomorphism in Hochschild cohomology. Here X_j is the restriction of X to an $\bar{\mathfrak{e}}_{\{j\}} - \bar{\mathfrak{e}}_\varnothing$ bimodule. It is easy to see that X_i is the bimodule associated to the inclusion $i_{\{j\},\varnothing *} : \bar{\mathfrak{e}}_{\{j\}} \longrightarrow \bar{\mathfrak{e}}_\varnothing$ (as introduced in Theorem 4.1.2.2). Thus in the above diagram $i_{\bar{\mathfrak{e}}_\varnothing}$ induces an isomorphism on Hochschild cohomology and we obtain a commutative diagram:

$$
\begin{array}{ccc}
\mathbf{C}(\bar{\mathfrak{e}}_\varnothing) & \xrightarrow{\cong} & \mathbf{C}(\bar{\mathfrak{e}}) \\
\| & & \downarrow \\
\mathbf{C}(\bar{\mathfrak{e}}_\varnothing) & \longrightarrow & \mathbf{C}(\bar{\mathfrak{e}}_{\{j\}})
\end{array}
$$

where now the lower and the rightmost maps are restriction maps. This finishes the proof. $\square$

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