Fourier-Mukai Transforms

Lutz Hille and Michel Van den Bergh

September 22, 2004

Abstract

In this paper we discuss some of the recent developments on derived equivalences in algebraic geometry.

Contents

1 Some background

In this paper we discuss some of the recent developments on derived equivalences in algebraic geometry but we don't intend to give any kind of comprehensive survey. It is better to regard this paper as a set of pointers to some of the recent literature.

To put the subject in context we start with some historical background. Derived (and triangulated) categories were introduced by Verdier in his thesis (see [56]) in order to simplify homological algebra. From this point of view the role of derived categories is purely technical.

The first pure algebro-geometric derived equivalence seems to appear in [38] where is it is shown that an abelian variety A and its dual $\hat{A}$ are derived equivalent. The equivalence resembles a Fourier-transform and is now known as a "Fourier-Mukai" transform.

In [4] Beilinson showed that $\mathbb{P}^n$ is derived equivalent to a (non-commutative) finite dimensional algebra. This explained earlier results by Barth and Hulek on the relation between vector bundles and linear algebra. Beilinson's result has been generalized to other varieties and has evolved into the theory of exceptional sequences (see for example [6]). The observation that derived equivalences do not preserve commutativity is significant for non-commutative algebraic geometry (see for example [22]).

Most algebraists probably became aware of the existence non-trivial derived equivalences when Happel showed that "tilting" (as introduced by Brenner and Butler [11]) leads to a derived equivalence between finite dimensional algebras [25]. This was generalized by Rickard who worked out the Morita theory for derived categories of rings [44, 45].

Hugely influential was the so-called homological mirror symmetry conjecture by Kontsevich [33] which states (very roughly) that for two Calabi-Yau manifolds X, Y in a mirror pair, the bounded derived category of coherent sheaves on X is equivalent to a certain triangulated category (the Fukaya category) related to the symplectic geometry of Y. The homological mirror symmetry conjecture was recently proven by Seidel for quartic surfaces (which are the simplest Calabi-Yau manifolds after elliptic curves) [49].

2 Notations and conventions

Throughout we work over the base field $\mathbb{C}$. The bounded derived category of coherent sheaves on a variety X is denoted by $\mathcal{D}^b(X)$. Similarly, the bounded derived category of finitely generated modules over an algebra A is denoted by $\mathcal{D}^b(A)$. The shift functor in the derived category is denoted by [1]. All functors between triangulated categories are additive and exact (i.e. they commute with shift and preserve distinguished triangles).

A sheaf is a coherent $\mathcal{O}_X$–module and a point in X is always a closed point. The structure sheaf of a point x will be denoted by $\mathcal{O}_x$. The canonical divisor of a smooth projective variety is denoted by K_X and the canonical sheaf is denoted by ω_X.

3 Basics on Fourier-Mukai transforms

Let X and Y be connected smooth projective varieties. We are interested in equivalences of the derived categories $\Phi : \mathcal{D}^b(Y) \longrightarrow \mathcal{D}^b(X)$. Such varieties X and Y are also called *Fourier-Mukai partners* and the equivalence Φ is called a *Fourier-Mukai transform*. In this section we will discuss some properties which remain invariant under Fourier-Mukai transforms. The main technical tool is Orlov's theorem (see below) which states that any derived equivalence $\Phi : \mathcal{D}^b(Y) \longrightarrow \mathcal{D}^b(X)$ is coming from a complex on the product $Y \times X$.

Given Fourier-Mukai X, Y it is also interesting to precisely classify the Fourier-Mukai transforms $\mathcal{D}^b(Y) \longrightarrow \mathcal{D}^b(X)$ (it is usually sufficient to consider $X = Y$). This is generally a much harder problem which has been solved in only a few special cases, notably abelian varieties [41] and varieties with ample canonical or anti-canonical divisor (see Theorem 4.4 below).

To start one has the following simple result.

Lemma 3.1 ([17, Lemma 2.1]). *If X and Y are Fourier-Mukai partners, then $\dim(X) = \dim(Y)$ and the canonical line bundles ω_X and ω_Y have the same order.*

Proof. The proof is an exercise in the use of *Serre functors* [10]. The Serre functor $S_X = - \otimes \omega_X[\dim(X)]$ on X is uniquely characterized by the existence of natural isomorphisms

$$\operatorname{Hom}_{\mathcal{D}^b(X)}(\mathcal{E}, \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{D}^b(X)}(\mathcal{F}, S_X\mathcal{E})^*. \tag{3.1}$$

By uniqueness it is clear that any Fourier-Mukai transform commutes with Serre functors. Pick a point $y \in Y$ and put $\mathcal{E} = \Phi(\mathcal{O}_y)$. The fact that $S_Y[-\dim Y](\mathcal{O}_y) \cong \mathcal{O}_y$ yields $S_X[-\dim Y](\mathcal{E}) \cong \mathcal{E}$, or $\mathcal{E} \otimes_X \omega_X[\dim X - \dim Y] \cong \mathcal{E}$. Looking at the homology of $\mathcal{E}$ we see that this impossible if $\dim Y \neq \dim X$. The statement about the orders of ω_X and ω_Y follows by considering the orders of the functors $S_X[-\dim X]$ and $S_Y[-\dim Y]$. $\square$

The following important result tells that any derived equivalence between $\mathcal{D}^b(Y)$ and $\mathcal{D}^b(X)$ is obtained from an object on the product $Y \times X$.

Theorem 3.2 ([40]). *Let $\Phi : \mathcal{D}^b(Y) \longrightarrow \mathcal{D}^b(X)$ be a fully faithful functor. Then there exists an object $\mathcal{P}$ in $\mathcal{D}^b(Y \times X)$, unique up to isomorphism, such that Φ is isomorphic to the functor*

$$\Phi^{\mathcal{P}}_{Y \to X}(-) := \pi_{X*}(\mathcal{P} \otimes_{\mathcal{O}_{Y \times X}} \pi_Y^*(-)),$$

where π_X and π_Y are the projection maps and π_{X*}, $\otimes$, and π_Y^* are the appropriate derived functors.

In the original statement of this theorem Φ was required to have a right adjoint but this condition is automatically fulfilled by [9, 10].

The object $\mathcal{P}$ in the theorem above is also called the *kernel* of the Fourier-Mukai transform.

Remark 3.3. *Theorem 3.2 is quite remarkable as for example its analogue for affine varieties or finite dimensional algebras is unknown (except for hereditary* *algebras [36]). Projectivity is used in the proof in the following way: let $\mathcal{L}$ be an ample line bundle on a projective variety X. Then for any coherent sheaf $\mathcal{F}$ on X one has $\mathrm{Hom}_{\mathrm{coh}(X)}(\mathcal{F}, \mathcal{L}^{-n}) = 0$ for large n. If X is for example affine then $\mathcal{O}_X$ is ample but this additional property does not hold.*

It would seem useful to generalize Theorem 3.2 to singular varieties, in particular those occurring in the minimal model program (see below). A first result in this direction has been obtained by Kawamata [29] who proves the analogue of Theorem 3.2 for orbifolds.

The real significance of Theorem 3.2 is that it makes it possible to define Φ on objects functorially derived from X and Y. For example (see [18, 42]) let $\mathrm{ch}'_X(-) = \mathrm{ch}_X(-).\,\mathrm{Td}(X)^{1/2}$ (where $\mathrm{ch}_X(-)$ is the Chern character and $\mathrm{Td}(X)$ is the Todd class of X). Using $\mathrm{ch}'_{Y\times X}(\mathcal{P})$ as kernel one finds a linear isomorphism of vector spaces

$$H^*(\Phi) : H^*(Y, \mathbb{Q}) \longrightarrow H^*(X, \mathbb{Q})$$

preserving parity of degree. Since the Chern character of $\mathcal{P}$ and the Todd class on $Y \times X$ may have denominators the same result is not a priory true for $H^*(X, \mathbb{Z})$. However it is true for elliptic curves (trivial) and for abelian and K3-surfaces [39].

Remark 3.4. *In order to circumvent the non-preservation of integrality it may be convenient to replace $H^*(X, \mathbb{Z})$ by topological K-theory [28] $K^*(X)^{\mathrm{top}} = K^0(X)^{\mathrm{top}} \oplus K^1(X)^{\mathrm{top}}$ which is the K-theory of complex vector bundles (not necessarily holomorphic) on the underlying real manifold of X. Topological K-theory is a cohomology theory satisfying the usual Eilenberg-Steenrod axioms except the dimension axiom (which fixes the cohomology of a point). Since $K^*(-)^{\mathrm{top}}$ has the appropriate functoriality properties [28] one proves that Φ induces an isomorphism*

$$K^*(\Phi)^{\mathrm{top}} : K^*(Y)^{\mathrm{top}} \to K^*(X)^{\mathrm{top}}$$

It follows from the Atiyah-Hirzebruch spectral sequence that $K^*(X)^{\mathrm{top}}$ is a finitely generated $\mathbb{Z}/2\mathbb{Z}$ graded abelian group such that the Chern-character

$$\mathrm{ch} : K^*(X)^{\mathrm{top}} \to H^*(X, \mathbb{Q})$$

induces an isomorphism [27, Eq (3.21)]

$$K^*(X)^{\mathrm{top}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong H^*(X, \mathbb{Q})$$

In good cases the lattices given by $K^*(X)^{\mathrm{top}}$ and $H^*(X, \mathbb{Z})$ are the same. This is for example the case for curves, K3 surfaces and abelian varieties.

By Riemann-Roch the following diagram is commutative

$$\begin{array}{ccc} K^0(Y) & \xrightarrow{K^0(\Phi)} & K^0(X) \\ \Big\downarrow{\scriptstyle \mathrm{ch}'_Y(-)} & & \Big\downarrow{\scriptstyle \mathrm{ch}'_X(-)} \\ H^*(Y, \mathbb{Q}) & \xrightarrow{H^*(\Phi)} & H^*(X, \mathbb{Q}) \end{array}$$

$K^0(X)$ is equipped with the so-called Euler form

$$e([E],[F]) = \sum_i (-)^i \dim \mathrm{Hom}_{\mathcal{D}^b(X)}(E, F[i])$$

which is of course preserved by $K^0(\Phi)$. The map $\mathrm{ch}'_X(-)$ is compatible with the Euler form up to sign provided one twist the standard bilinear form on cohomology (obtained from Poincare duality) slightly [18]. More precisely put

$$\check{v} = i^{\deg v} e^{-(1/2)K_X} v$$

and

$$\langle v, w\rangle = \deg(\check{v} \cup w)$$

Then

$$e([E],[F]) = -\langle \mathrm{ch}'_X(E), \mathrm{ch}'_X(F)\rangle$$

The map $H^i(\Phi)$ is an isometry for $\langle -,-\rangle$.

The standard grading on $H^*(X,\mathbb{C})$ is of course not preserved by a Fourier-Mukai transform. However there is a different grading with is preserved. Define

$${}^nH^*(X,\mathbb{C}) = \bigoplus_{j-i=n} H^{i,j}(X)$$

where $H^m(X,\mathbb{C}) = \oplus_{i+j=m} H^{i,j}(X,\mathbb{C}) = \oplus_{i+j=m} H^i(X,\Omega^j_X)$ is the Hodge decomposition [24, §0.6]. It is classical that algebraic cycles lie in ${}^0H^*(X,\mathbb{C})$. From the fact that the kernel of $H^*(\Phi)$ is algebraic it follows that $H^*(\Phi)$ preserves the ${}^*(-)$ grading.

As another application of functoriality note that if S is of finite type then there is an equivalence

$$\Phi_S : \mathcal{D}^b(Y_S) \to \mathcal{D}^b(X_S)$$

induced by $\mathcal{P}_S$ (i.e. a Fourier-Mukai transform extends to families).

Example 3.5. *Here we give an example of a Fourier-Mukai transform which is very important for mirror-symmetry. Assume first that Z is a four dimensional symplectic manifold and let $i : S^2 \to Z$ be an embedding of a sphere as a Lagrangian submanifold. Then there exists a symplectic automorphism τ of L which is trivial outside a tubular neighborhood of S^2 and which is the antipodal map on S^2 itself [50]. τ is called the* symplectic Dehn twist *of Z associated to i.*

By the homological mirror symmetry conjecture there should be an analogous notion for derived categories of varieties. This was worked out in [51]. It turns out that the analogue of a Lagrangian sphere is a so-called spherical object. To be more precise $\mathcal{E} \in \mathcal{D}^b(X)$ is spherical *if $\mathrm{Hom}^i_{\mathcal{D}^b(X)}(\mathcal{E},\mathcal{E})$ is equal to $\mathbb{C}$ for $i = 0, \dim X$ and is zero in all other degrees and if in addition $\mathcal{E} \cong \mathcal{E} \otimes \omega_X$.*

Associated to a spherical object $\mathcal{E} \in \mathcal{D}^b(X)$ there is an auto-equivalence $T_{\mathcal{E}}$ of $\mathcal{D}^b(X)$, informally defined by

$$T_{\mathcal{E}}(\mathcal{F}) = \mathrm{cone}\left(\mathrm{RHom}_{\mathcal{D}^b(X)}(\mathcal{E},\mathcal{F}) \otimes_{\mathbb{C}} \mathcal{E} \xrightarrow{\text{evaluation}} \mathcal{F}\right)$$

The non-functoriality of cones leads to a slight technical problem with the naturality of this definition. This would be a problem for abstract triangulated categories but it can be rectified here using the fact that $\mathcal{D}^b(X)$ (being a derived category) is the H^0-category of an exact DG-category.

It is easy to show that the kernel of $T_{\mathcal{E}}$ is given by

$$\operatorname{cone}\left(\check{\mathcal{E}} \boxtimes \mathcal{E} \xrightarrow{\phi} \mathcal{O}_\Delta\right)$$

where $\check{\mathcal{E}} = \mathrm{RHom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$, $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal and ϕ is the obvious map.

If X is a K3-surface then $\mathcal{O}_X$ is spherical and the kernel of $T_{\mathcal{O}_X}$ is given by $\mathcal{O}_X(-\Delta)$. Other examples of spherical objects are projective lines on smooth surfaces with self intersection -2 and restrictions of exceptional objects to anticanonical divisors. In particular this last construction yields spherical objects on hypersurfaces of degree $n+1$ in $\mathbb{P}^n$.

It is convenient to have a criterion for a functor of the form $\Phi^{\mathcal{P}}_{Y\to X}(-) := \pi_{X,*}(\mathcal{P} \otimes \pi_Y^*(-))$ to be an equivalence. The following result originally due to Bondal and Orlov [6] and slightly amplified by Bridgeland [14, Theorem 1.1] shows that we can use the skyscraper sheaves as test objects.

Theorem 3.6. *Let $\mathcal{P}$ be an object in $\mathcal{D}^b(Y \times X)$. Then the functor $\Phi := \Phi^{\mathcal{P}}_{Y\to X}(-) : \mathcal{D}^b(Y) \longrightarrow \mathcal{D}^b(X)$ is fully faithful if and only if the following conditions hold*

1. *for each point y in Y*

$$\mathrm{Hom}_{\mathcal{D}^b(X)}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)) = \mathbb{C}$$

2. *for each pair of points y_1 and y_2 and each integer i*

$$\mathrm{Hom}^i_{\mathcal{D}^b(X)}(\Phi(\mathcal{O}_{y_1}), \Phi(\mathcal{O}_{y_2})) = 0 \text{ unless } y_1 = y_2 \text{ and } 0 \le i \le \dim Y.$$

If these conditions hold then Φ is an equivalence if and only if $\Phi(\mathcal{O}_y) \otimes \omega_X \cong \Phi(\mathcal{O}_y)$ for all $y \in Y$.

Remark 3.7. *Assume that $\mathcal{P}$ is an object in $\mathrm{coh}(Y \times X)$ flat over Y and write $\mathcal{P}_y = \Phi(\mathcal{O}_y)$. Then the previous theorem implies that Φ is fully faithful if and only if*

1. *for each point y in Y*

$$\mathrm{Hom}_{\mathcal{D}^b(X)}(\mathcal{P}_y, \mathcal{P}_y) = \mathbb{C}$$

2. *for each pair of points $y_1 \neq y_2$ and each integer i*

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0$$

It is obvious that the conditions for Theorem 3.6 are necessary. Proving that they are also sufficient is much harder. Since the proof in [6] only works for equivalences between derived categories of coherent sheaves, we make explicit some of the steps in Bridgeland's proof (see [14]) which are valid for more general triangulated categories.

Let $\mathcal{A}$ be a triangulated category. A subset Ω is called *spanning* if for each object a in $\mathcal{A}$ each of the following conditions implies $a = 0$:

1. $\mathrm{Hom}^i(a, b) = 0$ for all $b \in \Omega$ and all $i \in \mathbb{Z}$,
2. $\mathrm{Hom}^i(b, a) = 0$ for all $b \in \Omega$ and all $i \in \mathbb{Z}$.

It is easy to see that the set of all skyscraper sheaves on a smooth projective variety X is a spanning class for $\mathcal{D}^b(X)$. Note that a spanning class will not usually generate $\mathcal{A}$ in any reasonable sense.

Theorem 3.8 ([14, Theorem 2.3]). *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor between triangulated categories with left and right adjoint. Then F is fully faithful if and only if there exists a spanning class Ω for $\mathcal{A}$ such that for all elements a_1, a_2 in Ω, and all integers i, the homomorphism*

$$F : \mathrm{Hom}^i_{\mathcal{A}}(a_1, a_2) \longrightarrow \mathrm{Hom}^i_{\mathcal{B}}(Fa_1, Fa_2)$$

is an isomorphism.

Recall that a category is called indecomposable if it is not the direct sum of two non-trivial subcategories. The derived category $\mathcal{D}^b(X)$ is indecomposable for X connected. For a finite dimensional algebra A the derived category $\mathcal{D}^b(A)$ is connected precisely when A is connected.

Theorem 3.9 ([16, Theorem 2.3]). *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful functor between triangulated categories with Serre functors $S_{\mathcal{A}}$, $S_{\mathcal{B}}$ (see (3.1)) possessing a left adjoint. Suppose that $\mathcal{A}$ is non-trivial and $\mathcal{B}$ is indecomposable. Let Ω be a spanning class for $\mathcal{A}$ and assume that $FS_{\mathcal{A}}(\omega) \cong S_{\mathcal{B}}F(\omega)$ for all $\omega \in \Omega$. Then F is a equivalence of categories.*

It follows from [9, 10] that $\Phi^{\mathcal{P}}_{Y \to X}$ has both a right and a left adjoint. Explicit formulas are for the left and the right adjoint are [14, Lemma 4.5]:

$$\Phi^{\check{\mathcal{P}} \otimes \pi_X^* \omega_X [\dim X]}_{X \to Y}(-) \quad \text{and} \quad \Phi^{\check{\mathcal{P}} \otimes \pi_Y^* \omega_Y [\dim Y]}_{X \to Y}(-)$$

Applying Theorems 3.8,3.9 with $F = \Phi^{\mathcal{P}}_{Y \to X}$ and $\Omega = \{\mathcal{O}_y \mid y \in Y\}$ almost proves Theorem 3.6 except that we seem to need additional information on $\mathrm{Hom}^i_{\mathcal{D}^b(X)}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y))$ for $i > 0$. It is not at all obvious but it turns out that this extra information is unnecessary. Although it is not clear how to formalize it, it seems that this part of the proof may generalize whenever Y is the solution of some type of moduli problem in a triangulated category $\mathcal{B}$ (with $\mathcal{P}$ being the universal family). See [15, 16, 54] for other manifestations of this principle.

4 The reconstruction theorem

It is quite trivial to reconstruct X from the abelian category $\mathrm{coh}(X)$. For example the points of X are in one-one correspondence with the objects in $\mathrm{coh}(X)$ without proper subobjects. With a little more work one can also recover the Zariski topology on X as well as the structure sheaf.

It is similarly of interest to know to which extent one can reconstruct a variety from its derived category. The existence of non-isomorphic Fourier-Mukai partners shows that this cannot be done in general, but it is possible if the canonical sheaf or the anticanonical sheaf is ample.

Theorem 4.1 ([7, Theorem 2.5]). *Let X be a smooth connected projective variety with either ω_X ample or ω_X^{-1} ample. Assume $\mathcal{D}^b(X)$ is equivalent to $\mathcal{D}^b(Y)$. Then X is isomorphic to Y.*

Proof. We give a proof based on Orlov's theorem. Note that Y is also connected since $\mathcal{D}^b(Y) \cong \mathcal{D}^b(X)$ is connected.

Let $\Phi : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ be the derived equivalence and let S be the Serre functor $- \otimes \omega_X[\dim X]$ on X. Recall that it is intrinsically defined by (3.1). We say that E in $\mathcal{D}^b(X)$ is a *point object* if

1. $E \cong S(E)[i]$ for some integer i,
2. $\mathrm{Hom}^i(E, E) = 0$ for all $i < 0$, and
3. $\mathrm{Hom}(E, E) = \mathbb{C}$.

It is easy to prove that the only point objects in $\mathcal{D}^b(X)$ (under the assumptions on ω_X) are the shifts of the skyscraper sheaves. The main point is 1., since this condition and the ampleness of $\omega_X^{\pm 1}$ easily implies that E has finite length cohomology.

It follows that Φ sends skyscraper sheaves to shifts of skyscraper sheaves. Then the proof may then be finished using Corollary 4.3 below. $\square$

We need the following standard fact.

Proposition 4.2. *Let $Z \to S$ be a flat morphism of schemes of finite type with S connected. Let $\mathcal{P} \in \mathcal{D}^-(\mathrm{coh}(Z))$ and and assume that for all $s \in S$ we have that $\mathcal{P} \overset{L}{\otimes}_{\mathcal{O}_Z} \pi^*\mathcal{O}_s \cong \mathcal{O}_z[n]$ for some $n \in \mathbb{Z}$, $z \in Z$. Then $\mathcal{P} \cong i_*\mathcal{L}[m]$ where $i : S \to Z$ is a section of π, $\mathcal{L} \in \mathrm{Pic}(S)$ and $m \in \mathbb{Z}$.*

Proof. We claim first that the support of the cohomology $\mathcal{P}$ is finite over S. Assume that this is false and let $H^i(\mathcal{P})$ be the highest cohomology group with non-finite support. Then, up to finite length sheaves we have $H^i(\mathcal{P}) \otimes_{\mathcal{O}_Z} \pi^*\mathcal{O}_s \cong H^i(\mathcal{P} \overset{L}{\otimes}_{\mathcal{O}_Z} \pi^*\mathcal{O}_s)$. Hence $H^i(\mathcal{P}) \otimes_{\mathcal{O}_Z} \pi^*\mathcal{O}_s$ has finite length for all s which is a contradiction.

It is now sufficient to prove that $\mathcal{P}_0 = \pi_*(\mathcal{P})$ is a shifted line bundle given that $\mathcal{P}_0 \overset{L}{\otimes}_{\mathcal{O}_S} \mathcal{O}_s$ has one-dimensional cohomology for all s.

Fix $s \in S$ and assume $\mathcal{P}_0 \overset{L}{\otimes}_{\mathcal{O}_S} \mathcal{O}_s \cong \mathcal{O}_s[n]$. Using Nakayama's lemma we deduce that there is a neighborhood U of s such that $H^i(\mathcal{P}_0 \mid U) = 0$ for $i > -n$. We temporarily replace S by U.

Applying $- \overset{L}{\otimes}_{\mathcal{O}_S} \mathcal{O}_s$ to the triangle

$$\tau_{\leq -n-1}\mathcal{P}_0 \to \mathcal{P}_0 \to H^{-n}(\mathcal{P}_0)[n] \to$$

we find $H^{-n}(\mathcal{P}_0) \otimes_{\mathcal{O}_S} \mathcal{O}_s \cong \mathcal{O}_s$ and $\mathrm{Tor}_1^{\mathcal{O}_S}(H^{-n}(\mathcal{P}_0), \mathcal{O}_s) = 0$. Hence $H^{-n}(\mathcal{P})$ is a line bundle on a neighborhood of s. Shrinking S further we may assume $\mathcal{P}_0 \cong \tau_{\leq -n-1}\mathcal{P}_0 \oplus H^{-n}(\mathcal{P}_0)[n]$ and hence $\tau_{\leq -n-1}\mathcal{P}_0 \overset{L}{\otimes}_{\mathcal{O}_S} \mathcal{O}_s = 0$. Shrinking S once again we have $\tau_{\leq -n-1}\mathcal{P}_0 = 0$ and thus $\mathcal{P}_0 \cong H^0(\mathcal{P}_0)[n]$ is a line bundle on a neighborhood of s.

Since this works for any s and S is connected we easily deduce that $\mathcal{P}_0$ is itself a shifted line bundle. □

We deduce the following

Corollary 4.3. *Assume that $\Phi : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ is a Fourier-Mukai transform between smooth connected projective varieties which sends skyscraper sheaves to shifted skyscraper sheaves. Then Φ is of the form $\sigma_*(- \otimes_{\mathcal{O}_X} \mathcal{L})[n]$ for an isomorphism $\sigma : Y \to X$, $\mathcal{L} \in \mathrm{Pic}(Y)$ and $n \in \mathbb{Z}$.*

Proof. By Proposition 4.2 the kernel of Φ must be of the form $\mathcal{P} = (1, \sigma_*)_*\mathcal{L}[n]$ for some map $\sigma : Y \to X$. The resulting $\Phi^{\mathcal{P}}_{Y \to X} = \sigma_*(- \otimes_{\mathcal{O}_X})[n]$ will be a derived equivalence if and only if σ is an isomorphism. □

One also obtains as a corollary the following result.

Theorem 4.4 ([7, Theorem 3.1]). *Let X be a smooth connected projective variety with ample canonical or anticanonical sheaf. Then the group of isomorphism classes of auto-equivalences of $\mathcal{D}^b(X)$ is generated by the automorphisms of X, the twists by line bundles and the translations.*

Remark 4.5. *It is clear that the notion of point object make sense for arbitrary triangulated categories with Serre functor.*

Let $\mathcal{D}$ be the bounded derived category of modules over a connected finite dimensional hereditary $\mathbb{C}$–algebra A. Then point objects only exist for A tame (or in the trivial case $A \cong \mathbb{C}$). In this case the point objects are the shifts of quasi-simple modules in homogeneous tubes. Let A be not necessary hereditary and we assume $\mathcal{D}^b(A)$ is equivalent to $\mathcal{D}^b(X)$ for some smooth projective variety X. Then $\mathcal{D}^b(A)$ has point objects. The situation is similar if we replace X by a weighted projective variety. However, it is an open problem to construct algebras A having (sufficiently many) point objects without knowing such an equivalence between $\mathcal{D}^b(A)$ and $\mathcal{D}^b(X)$ for some (weighted) projective variety X.

Note that there is a subtle point in the statement of Theorem 4.1. One does not *apriori* require Y to have ample canonical or anti-canonical divisor. If we preimpose this condition then Theorem 4.1 also follows from Theorem 4.6 below which morally corresponds to the fact that derived equivalences commute with Serre functors.

Theorem 4.6 ([42]). *Let X be a smooth projective variety. Then the integers $\dim \Gamma(X, \omega_X^{\otimes m})$ as well as the the canonical and anti-canonical rings are derived invariants.*

Assume that X is connected. For a Cartier divisor D denote by $R(X, D)$ the ring

$$R(X, D) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nD))$$

and by $K(X, D)$ the part of degree zero of the graded quotient field of $R(X, D)$. We have $K(X, D) \subset K(X)$ where $K(X)$ is the function field of X. By [53, Prop 5.7] $K(X, D)$ is algebraically closed in $K(X)$. If some positive multiple of D is effective then the D-Kodaira dimension $\kappa(X, D)$ of $K(X, D)$ is the transcendence degree of $K(X, D)$, otherwise we set $k(X, D) = -\infty$. It is clear that we have

$$\kappa(X, D) \leq \dim X$$

and in case of equality we have $K(X) = K(X, D)$.

The Kodaira dimension $\kappa(X)$ of X is $\kappa(X, K_X)$. X is of general type if $\kappa(X, K_X) = \dim X$.

Corollary 4.7 ([30, Theorem 2.3]). *The Kodaira dimension is invariant under Fourier-Mukai transforms. If X is of general type then any Fourier-Mukai partner of X is birational to X.*

Proof. This follows directly from Theorem 4.6 and the preceding discussion. □

5 Curves and surfaces

In this section we consider Fourier-Mukai transforms for smooth projective curves and smooth projective surfaces. For curves the situation is rather trivial: only elliptic curves admit non-trivial Fourier-Mukai transforms $\mathcal{D}^b(C) \cong \mathcal{D}^b(D)$, and in that case the curves C and D must be isomorphic. The group of auto-equivalences of $\mathcal{D}^b(C)$ is generated by the trivial ones and the classical Fourier-Mukai transform (which is almost the same as the auto-equivalence associated to the spherical object $\mathcal{O}_E$).

For surfaces the situation is more complicated and is worked out in detail in [17]. The classification of possible non-trivial Fourier-Mukai transforms is based on the classification of complex surfaces (see [2, page 188]). This classification is summarized in Table 1.

Class of X	$\kappa(X)$	n_X	$b_1(X)$	c_1^2	c_2
1) minimal rational surfaces	$-\infty$		0	$8, 9$	$4, 3$
3) ruled surfaces of genus $g \geq 1$	$-\infty$		$2g$	$8(1-g)$	$4(1-g)$
4)Enriques surfaces	0	2	0	0	12
5) hyperelliptic surfaces	0	$2, 3, 4, 6$	2	0	0
7) K3-surfaces	0	1	0	0	24
8) tori	0	1	4	0	0
9) minimal properly elliptic surfaces	1			0	≥ 0
10) minimal surfaces of general type	2	$\equiv 0 \bmod 2$	2	> 0	> 0

Table 1. Classification of algebraic smooth complex surfaces

Let us start with the case of curves. Let C be a smooth projective curve and denote by g_C the genus of C. According to the degree of the canonical divisor K_C there are three distinct classes:

1. $K_C < 0$: C is the projective line $\mathbb{P}^1(\mathbb{C})$ and $g_C = 0$,
2. $K_C = 0$: C is an elliptic curve and $g_C = 1$,
3. $K_C > 0$: C is a curve of general type and $g_C > 1$.

Using the reconstruction theorems 4.1 and 4.4 it is obvious that non-trivial Fourier-Mukai transforms can only exist for elliptic curves since K_C^{-1} is ample in case 1. and K_C is ample in case 3..

We will now look in somewhat more detail at the interesting case of elliptic curves. Note that if C, D are abelian varieties then it is known precisely when C and D are derived equivalent and furthermore the group $\mathrm{Aut}(\mathcal{D}^b(C))$ consisting of auto-equivalences of $\mathcal{D}^b(C)$ (up to isomorphism) is also completely understood [41]. Here we give an elementary account of the one-dimensional case. This is well-known and was explained to us by Tom Bridgeland. First we have the following result.

Theorem 5.1. *If C, D are derived equivalent elliptic curves then $C \cong D$.*

Proof. By the discussion in §3 the Hodge structures on $H^1(C, \mathbb{C})$ and $H^1(D, \mathbb{C})$ are isomorphic. Since the isomorphism class of an elliptic curve is encoded in its Hodge structure on $H^1(-, \mathbb{C})$ we are done. □

Determining the structure of $\mathrm{Aut}(\mathcal{D}^b(C))$ requires slightly more work. For an elliptic curve C let e_C be the Euler form on $K^0(C)$. By Serre duality e_C is skew symmetric. Put $\mathcal{N}(C) = K^0(C)/\operatorname{rad} e_C \cong \mathbb{Z}^2$. e_C defines a non-degenerate skew symmetric form (i.e. a symplectic form) on $\mathcal{N}(C)$ which we denote by the same symbol.

$\mathcal{N}(C)$ has a canonical basis given by $v_1 = [\mathcal{O}_C]$, $v_2 = [\mathcal{O}_x]$ ($x \in C$ arbitrary). The matrix of $e_C(v_i, v_j)_{ij}$ with respect to this basis is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

With respect to the standard basis the group of symplectic automorphisms of $\mathcal{N}(C)$ may be identified with $\mathrm{Sl}_2(\mathbb{Z})$.

Let T_1, T_2 be the auto-equivalences of C associated to the spherical objects $\mathcal{O}_C$ and $\mathcal{O}_x$. It is not hard to see that $T_2 = - \otimes_{\mathcal{O}_C} \mathcal{O}_C(x)$ so only T_1 is a non-trivial Fourier-Mukai transform.

One computes that with respect to the standard basis the action of T_1, T_2 on $\mathcal{N}(C)$ is given by matrices

$$T_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

These matrices are standard generators for $\mathrm{Sl}_2(\mathbb{Z})$ which satisfy the braid relation

$$T_1 T_2 T_1 = T_2 T_1 T_2 \tag{5.1}$$

Remark 5.2. *Since the objects $\mathcal{O}_C$, $\mathcal{O}_x$ form a so-called A_2 configuration [51] the relation* (5.1) *actually holds in* $\mathrm{Aut}(\mathcal{D}^b(C))$.

We have:

Theorem 5.3. *Let $\mathrm{Aut}^0(\mathcal{D}^b(C))$ be the subgroup of $\mathrm{Aut}(\mathcal{D}^b(C))$ consisting of auto-equivalences of the form $\sigma_*(- \otimes_{\mathcal{O}_C} \mathcal{L})[n]$ where $\sigma \in \mathrm{Aut}(C)$, $\mathcal{L} \in \mathrm{Pic}^0(C)$ and $n \in 2\mathbb{Z}$. Then the symplectic action of $\mathrm{Aut}(\mathcal{D}^b(C))$ on $\mathcal{N}(C)$ yields an exact sequence*

$$0 \to \mathrm{Aut}^0(\mathcal{D}^b(C)) \to \mathrm{Aut}(\mathcal{D}^b(C)) \to \mathrm{Sl}_2(\mathbb{Z}) \to 0.$$

Proof. The existence of T_1, T_2 implies that the map $\mathrm{Aut}(\mathcal{D}^b(C)) \to \mathrm{Sl}_2(\mathbb{Z})$ is onto.

Assume that $\Phi \in \mathrm{Aut}(\mathcal{D}(C))$ act trivially on $\mathcal{N}(C)$. It is easy to see that for an object $\mathcal{E} \in \mathcal{D}^b(C)$ this implies

$$\begin{aligned} \deg \Phi(\mathcal{E}) &= \deg \mathcal{E} \\ \operatorname{rk} \Phi(\mathcal{E}) &= \operatorname{rk} \mathcal{E} \end{aligned} \tag{5.2}$$

The abelian category $\mathrm{coh}(D)$ is hereditary and hence every object in $\mathcal{D}^b(D)$ is the direct sum of its cohomology. Since $\Phi(\mathcal{O}_y)$ must be indecomposable we deduce from (5.2) that $\Phi(\mathcal{O}_y)$ is a twisted skyscraper sheaf.

We find by Corollary 4.3 that $\Phi = \sigma_*(- \otimes_{\mathcal{O}_C} \mathcal{L})[n]$. The fact that Φ acts trivially on $\mathcal{N}(C)$ implies $\deg \mathcal{L} = 0$ and n is even. $\square$

Remark 5.4. *Using similar arguments as above it is easy to see that the orbits of the action $\mathrm{Aut}(\mathcal{D}^b(C))$ on the indecomposable objects in $\mathcal{D}^b(C)$ are indexed by $\mathbb{N} \setminus \{0\}$. The quotient map is given by*

$$E \mapsto \gcd(\mathrm{rk}(E), \deg(E))$$

In particular any indecomposable vector bundle is in the orbit of an indecomposable finite length sheaf.

Remark 5.5. *The situation for elliptic curves is very similar to the situation for tubular algebras [46, 26], tubular canonical algebras, or tubular weighted projective curves (weighted projective curves of genus one) [35]. We quickly explain how these three categories $\mathcal{D}^b(C)$ (C an elliptic curve), $\mathcal{D}^b(\mathbb{X})$ ($\mathbb{X}$ a tubular weighted projective curve) and $\mathcal{D}^b(\Lambda)$ (Λ a tubular canonical algebra or a tubular algebra) are related to each other. Any elliptic curve C admits a non-trivial automorphism $\phi : C \longrightarrow C$ $x \mapsto -x$. Let $G \cong \mathbb{Z}/2\mathbb{Z}$, generated by ϕ. The category of G-equivariant sheaves on C is isomorphic to the category of coherent sheaves on a weighted projective line of type $\mathbb{D}_4$. For the remaining types $\mathbb{E}_{6,7,8}$ we consider elliptic curves with complex multiplication of order 3, 4 or 6, respectively. Then an analogous result holds for those curves (see also [48]).*

Now we discuss the case of surfaces. In the rest of this section a surface will be a smooth projective surface.

Remember that a surface X is called minimal if it does not contain an exceptional curve C (i.e. a smooth rational curve with self intersection -1). The possible non-trivial Fourier-Mukai partners for minimal surfaces were classified by Bridgeland and Maciocia in [17]. This classification is based on the classification of surfaces (see [2, page 188]) as summarized in Table 1 (we have only listed the algebraic surfaces as these are the only ones of interest to us).

Table 1 is in terms of some standard invariants which we first describe. We have already mentioned the Kodaira dimension $\kappa(X)$. It is either $-\infty, 0, 1$ or 2 and divides the minimal surfaces into four classes. For an arbitrary surface X there is always a map $X \to X_0$ to a minimal surface. If $k(X) \geq 0$ then X_0 depends only on the birational equivalence class of X [2, Proposition (4.6)].

Further invariants are the first Betti number $b_1(X) = \dim H^1(X, \mathbb{C})$, the square of the first Chern class $c_1^2(X) = K_X^2$ and the second Chern class $c_2(X)$ (where $c_i = c_i(T_X)$). Finally, for surfaces of Kodaira dimension zero one also needs the smallest natural number n_X with $n_X K_X = 0$.

The invariants $b_1(X)$, $c_1(X)^2$, $c_2(X)$ contain exactly the same information as the (numeric) Hodge diamond of X:

$$\begin{array}{ccccc} & & 1 & & \\ & q(X) & & q(X) & \\ p_g(X) & & h^{1,1}(X) & & p_g(X) \\ & q(X) & & q(X) & \\ & & 1 & & \end{array}$$

where $p_g(X)$ is the geometric genus of X, $q(X)$ is the Noether number of X and $h^{ij}(X) = \dim H^{ij}(X, \mathbb{C})$. One has

$$\begin{aligned} b_1(X) &= 2q(x) \\ c_2(X) &= 2 + 2p_g(X) - 4q(X) + h^{1,1}(X) \\ \frac{1}{12}(c_1(X)^2 + c_2(X)) &= 1 - q(x) + p_g(X)) \end{aligned}$$

The second line is the Gauss-Bonnet formula [24, §3.3] which says that $c_2(X)$ is equal to the Euler number $\sum_i \dim(-1)^i \dim H^i(X, \mathbb{C})$ of X. The third formula is Noether's formula. It follows from applying the Riemann-Roch theorem [2, Thm I.(5.3)] to the structure sheaf.

For abelian and K3-surfaces the so-called transcendental lattice is of interest. First note that $H^2(X, \mathbb{Z})$ is free. For abelian surfaces this is clear since they are tori and for K3 surfaces it is [2, Prop VIII(3.2)]. The Neron-Severi lattice is $N_X = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ and the transcendental lattice T_X is the sublattice of $H^2(X, \mathbb{Z})$ orthogonal to S_X.

Theorem 5.6 ([17, Theorem 1.1]). *Let X and Y be a non-isomorphic smooth connected complex projective surfaces with equivalent derived categories $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$ such that X is minimal. Then either*

1. *X is a torus (an abelian surface, in class 8)) and Y is also a torus with Hodge-isometric transcendental lattice,*
2. *X is a K3-surface (a surface in class 7)) and Y is also a K3-surface with Hodge isometric transcendental lattice, or*
3. *X is an elliptic surface and Y is another elliptic surface obtained by taking a relative Picard scheme of the elliptic fibration on X.*

A Hodge isometry between transcendental lattices is an isometry under which the one dimensional subspaces $H^0(X, \omega_X)$ and $H^0(Y, \omega_Y)$ of $T_X \otimes_{\mathbb{R}} \mathbb{C}$ and $T_Y \otimes_{\mathbb{R}} \mathbb{C}$ correspond.

The proof of Theorem 5.6 is quite involved and uses case by case analysis quite essentially. As a very rough indication of some of the methods one might use, let us show that if X is minimal then so is Y and they are in the same class. Along the way we will settle the easy case $\kappa(X) = 2$.

Step 1: By Corollary 4.7 and the discussion in §3 X and Y have the same Kodaira dimension and the same Hodge diamond. In particular they have the same $b_1(-)$, $c_1(-)^2$ and $c_2(-)$. Hence if they are both minimal then they are in the same class.

Step 2: Assume now that X is minimal and let $Y \to Y_0$ be a minimal model of Y. We have $b_1(Y) = b_1(Y_0)$ [2, Theorem I.(9.1)]. If $\kappa(X) = -\infty, 1, 2$ then the class of X is recognizable from $b_1(X)$ and hence Y_0 must be in the same class as X. If Y_0 is not in class 1,10) then it follows from the classification that $c_1(Y_0)^2 = c(X)^2$ and hence $c_1(Y_0)^2 = c_1(Y)^2$. If Y_0 is in class 10) then by Corollary 4.7 we have $X = Y_0$ and hence we also have $c_1(Y_0)^2 = c_1(Y)^2$. Since $c_1(-)^2$ changes by one under a blowup [2, Theorem I.(9.1)(vii)] it follows in these cases that $Y = Y_0$.

If Y_0 is is in class 1) then in principle we could have $c_1(Y_0)^2 = 9$, $c_1(Y)^2 = c_1(X)^2 = 8$. But then in Y is the blowup of $\mathbb{P}^2$ in a point and hence is Del-Pezzo. We conclude by the reconstruction theorem 4.1 that $X = Y$ which is a contradiction.

Step 3: If $\kappa(X) = 0$ then ω_X has finite order and hence the same is true for Y by Lemma 3.1. This is impossible if Y is not minimal.

Let us also say a bit more on the K3 and abelian case. Assume that X is a a K3 or abelian surface. Then according [39] the Chern character $K^0(X) \to H^*(X, \mathbb{Q})$ takes it values in $H^*(X, \mathbb{Z})$. As before let $\mathcal{N}(X)$ be $K^0(X)$ modulo the radical of the Euler form. Since the intersection form on $H^*(X, \mathbb{Z})$ is non-degenerate it follows that $\mathcal{N}(X)$ is the image of $K^0(X)$ in $H^*(X, \mathbb{Z})$. It is easy to see that the orthogonal to $\mathcal{N}(X)$ is T_X.

Now assume that X and Y are derived equivalent K3 or abelian surfaces. Again by [39] the induced isometry between $H^*(X, \mathbb{Q})$ and $H^*(Y, \mathbb{Q})$ yields an isometry between $H^*(X, \mathbb{Z})$ and $H^*(X, \mathbb{Z})$. By the above discussion there is an isometry between T_X and T_Y. This is a Hodge isometry since $H^0(X, \omega_X) = {}^2H^*(X, \mathbb{C})$.

The complete result for K3 or abelian surfaces is as follows.

Theorem 5.7 ([40], see also [17]). *Let X and Y be a pair of either K3-surfaces or abelian surfaces (tori) then the following statements are equivalent.*

1. *There exists a Fourier-Mukai transform $\Phi : \mathcal{D}^b(Y) \longrightarrow \mathcal{D}^b(X)$.*
2. *There is an Hodge isometry $\phi^t : T(Y) \longrightarrow T(X)$.*
3. *There is an Hodge isometry $\phi : H^{2*}(Y, \mathbb{Z}) \longrightarrow H^{2*}(X, \mathbb{Z})$.*
4. *Y is isomorphic to a fine, two-dimensional moduli space of stable sheaves on X.*

The non minimal case is covered by the following result of Kawamata.

Theorem 5.8 ([30, Theorem 1.6]). *Assume that X, Y are Fourier-Mukai partners but with X not minimal. Then there are only a finite number of possibilities for Y (as in the minimal case). If X is not isomorphic to a relatively minimal elliptic rational surface then X and Y are isomorphic.*

It remains to classify the auto-equivalences of the derived category $\mathcal{D}^b(X)$ for a surface X. Orlov solved this problem for an abelian surface [41] (and more generally for abelian varieties). The most interesting open case is given by K3-surfaces although here important progress has recently been made by Bridgeland [12, 13]. For any X Bridgeland constructs a finite dimensional complex manifold $\mathrm{Stab}(X)$ on which $\mathrm{Aut}(D^b(X))$ acts naturally. Roughly speaking the points of $\mathrm{Stab}(X)$ correspond to t-structures on $D^b(X)$ together with extra data defining Harder-Narasimhan filtrations on objects in the heart. The definition of $\mathrm{Stab}(X)$ was directly inspired by work of Michael Douglas on stability in string theory [21]. It seems very important to obtain a better understanding of the space $\mathrm{Stab}(X)$.

6 Threefolds and higher dimensional varieties

If X is a projective smooth threefolds then just as in the surface case one would like to find a unique smooth minimal X_0 birationally equivalent to X. Unfortunately it is well known that this is not possible so some modifications have to be made. In particular one has to allow X_0 to have some mild singularities, and furthermore X_0 will in general be far from unique.

Throughout all our varieties are projective. We say that X is *minimal* if X is $\mathbb{Q}$-Gorenstein and K_X is numerically effective. I.e. for any curve $C \subset X$ we have $K_X \cdot C \geq 0$.

A natural category to work in are varieties with *terminal* singularities. Recall that a projective variety X has terminal singularities if it is $\mathbb{Q}$-Gorenstein and for a (any) resolution $f : Z \to X$ the discrepancy ($\mathbb{Q}$-)divisor $K_Z - f^*K_X$ contains every exceptional divisor with strictly positive coefficients. If $\dim X \leq 2$ and X has terminal singularities then X is smooth. So terminal singularities are indeed very mild.

If X is a threefold with terminal singularities then there exists a map $f : Z \to X$ which is an isomorphism in codimension one such that Z terminal, and $\mathbb{Q}$-factorial [32, Theorem 6.25]. Minimal threefolds with $\mathbb{Q}$-factorial terminal singularities are the "end products" of the three dimensional minimal program. Such minimal models are however not unique. One has the following classical result by Kollar [31].

Theorem 6.1. *Any birational map between minimal threefolds with $\mathbb{Q}$-factorial terminal singularities can be decomposed as a sequence of* flops*.*

Recall that a flop is a birational map which factors as $(f^+)^{-1}f$

$$\begin{tikzcd} X \arrow[rr, dotted] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & W & \end{tikzcd}$$

where f, f^+ are isomorphisms in codimension one such that K_X and K_{X^+} are $\mathbb{Q}$-trivial on the fibers of f and f^+ respectively and such that there is a $\mathbb{Q}$-Cartier divisor D on X with the property that D is relatively ample for f and $-D$ is relatively ample for f^+.

Example 6.2. *The easiest (local) example of a flop is the Atiyah flop [43]: Let $W = Spec\,(\mathbb{C}[x,y,z,u]/(xu - yz)$ be the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1$ associated to the line bundle $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$. W has an isolated singularity in the origin which may be resolved in two different ways $X \longrightarrow W \longleftrightarrow X^+$ by blowing up the ideals (x,y) and (x,z). The varieties X and X^+ are related by a flop.*

How does one construct a minimal model? Assume that X has $\mathbb{Q}$-factorial terminal singularities such that K_X is not numerically effective. The celebrated cone theorem [19, 32] allows one to construct a map $f : X \to W$ with relatively ample $-K_X$ such that one of the following properties holds [19, Thm (5.9)]

1. $\dim X > \dim W$ and f is a $\mathbb{Q}$-Fano fibration.
2. f is birational and contracts a divisor.
3. f is birational and contracts a subvariety of codimension ≥ 2.

Case 1. is what one would get by applying the cone theorem to $\mathbb{P}^2$. The result would be the contraction $\mathbb{P}^2 \to \mathrm{pt}$. In the case of surfaces 2. corresponds to blowing down exceptional curves. In general the result is again a variety with terminal singularities and smaller Neron-Severi group. Case 3. represents an new phenomenon which only occurs in dimension three and higher. In this case W may be not be $\mathbb{Q}$-Gorenstein so one is out of the category one wants to work in. In order to continue at this point one introduces a new operation called a *flip*. A flip is a birational map which factors as $(f^+)^{-1}f$

$$\begin{tikzcd} X \arrow[rr, dotted] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & W & \end{tikzcd}$$

where f, f^+ are isomorphisms in codimension one such that $-K_X$ is relatively ample for f, K_X is relatively ample for f^+ and X^+ again has $\mathbb{Q}$-factorial terminal singularities. The existence of three dimensional flips was settled by Mori in [37]. In higher dimension it is still open.

Example 6.3. *Let us give an easy example of a (higher dimensional) flip generalizing Example 6.2. Let W be the affine cone over $\mathbb{P}^m \times \mathbb{P}^n$ ($m \leq n$) associated to the line bundle $\mathcal{O}_{\mathbb{P}^m\times\mathbb{P}^n}(1,1)$. W has two canonical resolutions, the first one X being given as the total space of the vector bundle $\mathcal{O}(1)^{\oplus n}$ over $\mathbb{P}^m$ and the second one X^+ as the total space of the vector bundle $\mathcal{O}(1)^{\oplus m}$ over $\mathbb{P}^n$. The birational map $X \dashrightarrow X^+$ is a flip.*

Following (and slightly generalizing) [8] (see also [30]) let us say that a birational map $X \dashrightarrow X^+$ between $\mathbb{Q}$-Gorenstein varieties is a *generalized flip* if there is a commutative diagram with $\tilde{X}$ smooth

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\pi^+"] & \\ X \arrow[rr, dotted] & & X^+ \end{tikzcd}$$

such that $D = \pi^* K_X - {\pi^+}^*(K_{X^+})$ is effective. If $D = 0$ then $X \dashrightarrow X^+$ is a generalized flop.

Bondal and Orlov [8] state the following conjecture (see also [30]).

Conjecture 6.4. *For any generalized flip $X \dashrightarrow X^+$ between smooth projective varieties there is a full faithful functor $D^b(X^+) \to D^b(X)$. This functor is an equivalence for generalized flops.*

One could think of this conjecture as the foundation for a "derived minimal model" program.

As evidence of the fact that smooth projective varieties related by a generalized flop are expected to have many properties in common we recall the following very general result by Batyrev and Kontsevich.

Theorem 6.5. *If X and X^+ smooth varieties related by a generalized flop then they have the same Hodge numbers.*

Proof. (see [20]) If X and X^+ are related by a generalized flop then they have the same "stringy E-function". Since X and X^+ are smooth the stringy E-function is equal to usual E-function which encodes the Hodge numbers. $\square$

Remark 6.6. *The relation between Conjecture 6.4 and Theorem 6.5 seems rather subtle. Indeed a non-trivial Fourier-Mukai transform does not usually preserve cohomological degree and hence certainly does not preserve the Hodge decomposition.*

For non-smooth varieties $D^b(X)$ is probably not the correct object to consider. If X is $\mathbb{Q}$-Gorenstein then every point $x \in X$ has some neighborhood U_x such that on U_x there is some positive number m_x with the property $m_x K_x = 0$. Then K_x generates a cover $\tilde{U}_x$ of U_x on which $\mathbb{Z}/m\mathbb{Z}$ is acting naturally. Gluing the local quotient stacks $\tilde{U}_x/(\mathbb{Z}/m\mathbb{Z})$ defines a Deligne-Mumford stack [34] $\mathcal{X}$ birationally equivalent to X. As usual we write $D^b(\mathcal{X})$ for $D^b(\mathrm{coh}(\mathcal{X}))$. The following result summarizes what is currently known in dimension three concering the categories $D^b(\mathcal{X})$.

Theorem 6.7. *Let $\alpha : X \dashrightarrow X^+$ be a generalized flop between threefolds with $\mathbb{Q}$-factorial terminal singularities.*

1. *α is a composition of flops.*
2. *There is a corresponding equivalence $D^b(\mathcal{X}) \to D^b(\mathcal{X}^+)$.*

In this generality this result was proved by Kawamata in [30]. The corresponding result in the smooth case was first proved by Bridgeland in [15]. By 1) it is sufficient to consider the case of flops. While trying to understand Bridgeland's proof the second author produced a mildly different proof of the result [55]. Some of the ingredients in this new proof were adapted to the case of stacks by Kawamata.

Let us give some more comments on flips and flops. Flips and flops occur very naturally in invariant theory [52] and toric geometry and, as a particular case, for moduli spaces of thin sincere representations of quivers.

Batyrev's construction of Calabi-Yau varieties [3] uses toric geometry, in particular toric Fano varieties. Those varieties correspond to reflexive polytopes. Reflexive polytopes can also be constructed directly from quivers, however, this class of reflexive polytopes is very small. For moduli spaces of thin sincere quiver representations of dimension three all flips are actually flops.

Remark 6.8. *The results above should have consequences for derived categories of modules over finite dimensional algebras. However, no example is known of a derived equivalence between a bounded derived category $\mathcal{D}^b(A)$ of modules over finite dimensional algebra A and $\mathcal{D}^b(X)$, where X admits a flop. The "closest" examples to such an equivalence are the fully faithful functors constructed in [1]. If one allows flips (instead of flops) such equivalences exist, one may find toric varieties Y with a full strong exceptional sequence of line bundles. However, for its counterpart W under the flip such sequences are not known. Strongly related to this problem is a conjecture of A. King, that each smooth toric variety admits a full strong exceptional sequence of line bundles, however, even the existence of a full exceptional sequence of line bundles is an open problem.*

7 Non-commutative rings in algebraic geometry

In the previous section we considered mainly Fourier-Mukai transforms between algebraic varieties. There are also species of Fourier-Mukai transforms where one of the partners is non-commutative. In this section we discuss some examples. In contrast to the previous sections our algebraic varieties will not always be projective.

Let $f : X \to W$ be a projective birational map between Gorenstein varieties. f is said to be a crepant resolution if $\tilde{X}$ is smooth and if $f^*\omega_W = \omega_X$. A variant of Conjecture 6.4 is the following:

Conjecture 7.1. *Assume that W has Gorenstein singularities and that we have two crepant resolution.*

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & W & \end{tikzcd}$$

Then X and X^+ are derived equivalent.

We will now consider a mild non-commutative situation to which a similar conjecture applies. Let $G \subset \mathrm{Sl}_n(\mathbb{C})$ be a finite group and put $W = \mathbb{C}^n/G$. Write $D^b_G(\mathbb{C}^n)$ for the category of G equivariant coherent sheaves on $\mathbb{C}^n$ and let $X \to W$ be a crepant resolution W.

Conjecture 7.2. *$D^b(X)$ and $D^b_G(\mathbb{C}^n)$ are derived equivalent.*

If A is the skew group ring $\mathcal{O}(\mathbb{C}^n)*G$ then one may view A as a *non-commutative crepant resolution* of $\mathbb{C}^n/G$. Conjecture 7.2 may be reinterpreted as saying that all commutative crepant resolutions are derived equivalent to a non-commutative one. So in that sense it is an obvious generalization of Conjecture 7.1. A proper definition of a non-commutative crepant resolution together with a suitably generalized version of Conjecture 7.2 was given in [54]. An example where this generalized conjecture applies is [23]. A similar but slightly different conjecture is [8, Conjecture 5.1].

Conjecture 7.2 has now been proved in two cases. First let X be the irreducible component of the G-Hilbert scheme of $\mathbb{C}^n$ containing the regular representation. Then we have the celebrated BKR-theorem [16].

Theorem 7.3. *Assume that $\dim X \leq n+1$ (this holds in particular if $n \leq 3$). Then X is a crepant resolution of W and $D^b(X)$ is equivalent to $D^b_G(\mathbb{C}^n)$.*

Note that this theorem, besides establishing the expected derived equivalence, also produces a specific crepant resolution of W. For $n = 3$ this was done earlier by a case by case analysis (see [47] and the references therein).

Very recently the following result was proved.

Theorem 7.4. *[5] Assume that G acts symplectically on $\mathbb{C}^n$ (for some arbitrary linear symplectic form). Then Conjecture 7.2 is true.*

Somewhat surprisingly this result is proved by reduction to characteristic p.

Let us now discuss a similar but related problem. For a given scheme X one may want to find algebras A derived equivalent to X. One has the following very general result.

Theorem 7.5 ([9]). *Assume that X is separated. Then there exist a perfect complex E such that $D(\mathrm{Qcoh}(X))$ is equivalent to $D(A)$ where A is the DG-algebra $\mathrm{RHom}_{\mathcal{O}_X}(E,E)$.*

Recall that a perfect complex is one which is locally quasi-isomorphic to a finite complex of finite rank vector bundles.

In order to replace DG-algebras by real algebras let us say that a perfect complex $E \in D(\mathrm{Qcoh}(X))$ is *classical tilting* if it generates $D(\mathrm{Qcoh}(X))$ (in the sense that $\mathrm{RHom}_{\mathcal{O}_X}(E,U)=0$ implies $U=0$) and $\mathrm{Hom}^i_{\mathcal{O}_X}(E,E)=0$ for $i\neq 0$. One has the following result.

Theorem 7.6. *Assume that X is projective over a noetherian affine scheme of finite type and assume $E \in D(\mathrm{Qcoh}(X))$ is a classical tilting object. Put $A=\mathrm{End}_{\mathcal{O}_X}(E)$. Then*

1. $\mathrm{RHom}_{\mathcal{O}_X}(E,-)$ *induces an equivalence between $D(\mathrm{Qcoh}(X))$ and $D(A)$.*
2. *This equivalence restricts to an equivalence between $D^b(\mathrm{coh}(X))$ and $D^b(\mathrm{mod}(A))$.*
3. *If X is smooth then A has finite global dimension.*

Proof. 1) is just a variant on Theorem 7.5. The inverse functor is $-\overset{L}{\otimes}_A E$. To prove 2) note that the perfect complexes are precisely the compact objects (see [9, Theorem 3.1.1] for a very general version of this statement). Hence perfect complexes are preserved under $-\overset{L}{\otimes}_A E$. An object U has bounded cohomology if and only for any perfect complex C one has $\mathrm{Hom}(C,U[n])=0$ for $|n|\gg 0$. Hence objects with bounded cohomology are preserved as well. Now let Z be an object in $D^b(\mathrm{mod}(A))$. Then it easy to see that $\tau_{\geq n}(Z\overset{L}{\otimes}_A E)$ is in $D^b(\mathrm{coh}(X))$ for any n. Since $Z\overset{L}{\otimes}_A E$ has bounded cohomology we are done. To prove 3) note that for any $U,V\in \mathrm{mod}(A)$ we have $\mathrm{Ext}^i_A(U,V)$ for $i\gg 0$. Since A has finite type this implies that A has finite global dimension. $\square$

Classical tilting objects (and somewhat more generally: "exceptional collections") exist for many classical types of varieties [6]. The following somewhat abstract result was proved in [55].

Theorem 7.7. *Assume that $f: Y\to X$ is a projective map between varieties, with X affine such that $Rf_*\mathcal{O}_Y=\mathcal{O}_X$ and such that $\dim f^{-1}(x)\leq 1$ for all $x\in X$. Then Y has a classical tilting object.*

This result was inspired by Bridgeland's methods in [15]. It applies in particular to resolutions of three-dimensional Gorenstein terminal singularities. It also has a globalization if X is quasi-projective instead of affine.

References

[1] K. Altmann and L. Hille, *Strong exceptional sequences provided by quivers*, Algebr. Represent. Theory **2** (1999), no. 1, 1–17.

[2] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 4, Springer-Verlag, Berlin, 1984.

[3] V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom. **3** (1994), no. 3, 493–535.

[4] A. A. Beĭ linson, *The derived category of coherent sheaves on* $\mathbf{P}^n$, Selecta Math. Soviet. **3** (1983/84), no. 3, 233–237, Selected translations.

[5] R. Bezrukavnikov and D. Kaledin, *McKay equivalence for symplectic resolutions of singularities*, math.AG/0401002.

[6] A. Bondal and D. O. Orlov, *Semi-orthogonal decompositions for algebraic varieties*, math.AG 9506012.

[7] ———, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compositio Math. **125** (2001), no. 3, 327–344.

[8] ———, *Derived categories of coherent sheaves*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 47–56.

[9] A. Bondal and M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.

[10] A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1183–1205, 1337.

[11] S. Brenner and M. C. R. Butler, *Generalizations of the Bernstein-Gel′fand-Ponomarev reflection functors*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979) (Berlin), Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 103–169.

[12] T. Bridgeland, *Stability conditions on K3 surfaces*, math.AG/0307164.

[13] ———, *Stability conditions on triangulated categories*, math.AG/0212237.

[14] ———, *Equivalences of triangulated categories and Fourier-Mukai transforms*, Bull. London Math. Soc. **31** (1999), no. 1, 25–34.

[15] ———, *Flops and derived categories*, Invent. Math. **147** (2002), no. 3, 613–632.

[16] T. Bridgeland, A. King, and M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), no. 3, 535–554 (electronic).

[17] T. Bridgeland and A. Maciocia, *Complex surfaces with equivalent derived categories*, Math. Z. **236** (2001), no. 4, 677–697.

[18] A. Caldararu, *The Mukai pairing, II: the Hochschild-Kostant-Rosenberg isomorphism*, math.AG/0308080.

[19] H. Clemens, J. Kollár, and S. Mori, *Higher-dimensional complex geometry*, Astérisque (1988), no. 166, 144 pp. (1989).

[20] A. Craw, *An introduction to motivic integration*, math.AG/9911179.

[21] M. R. Douglas, *Dirichlet branes, homological mirror symmetry, and stability*, Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 395–408.

[22] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985) (Berlin), Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 265–297.

[23] I. Gordon and S. P. Smith, *Representations of symplectic reflection algebras and resolutions of deformations of symplectic quotient singularities*, math.RT/0310187.

[24] P. Griffiths and J. Harris, , Wiley Classics Library, John Wiley & Sons Inc., New York, 1994, Reprint of the 1978 original.

[25] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.

[26] D. Happel and C. M. Ringel, *The derived category of a tubular algebra*, Representation theory, I (Ottawa, Ont., 1984) (Berlin), Lecture Notes in Math., vol. 1177, Springer, Berlin, 1986, pp. 156–180.

[27] P. Hilton, *General cohomology theory and K-theory*, Course given at the University of São Paulo in the summer of 1968 under the auspices of the Instituto de Pesquisas Matemáticas, Universidade de São Paulo. London Mathematical Society Lecture Note Series, vol. 1, Cambridge University Press, London, 1971.

[28] M. Karoubi, *K-theory*, Springer-Verlag, Berlin, 1978, An introduction, Grundlehren der Mathematischen Wissenschaften, Band 226.

[29] Y. Kawamata, *Equivalences of derived catgories of sheaves on smooth stacks*, math.AG/0210439.

[30] ———, *D-equivalence and K-equivalence*, J. Differential Geom. **61** (2002), no. 1, 147–171.

[31] J. Kollár, *Flops*, Nagoya Math. J. **113** (1989), 15–36.

[32] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[33] M. Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 120–139.

[34] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 39, Springer-Verlag, Berlin, 2000.

[35] H. Lenzing and H. Meltzer, *Sheaves on a weighted projective line of genus one and representations of a tubular algebra*, Proceedings of the Sixth International Conference on Representations of Algebras (Ottawa, ON, 1992) (Ottawa, ON), Carleton-Ottawa Math. Lecture Note Ser., vol. 14, Carleton Univ., 1992, p. 25.

[36] J.-i. Miyachi and A. Yekutieli, *Derived Picard groups of finite-dimensional hereditary algebras*, Compositio Math. **129** (2001), no. 3, 341–368.

[37] S. Mori, *Flip theorem and the existence of minimal models for 3-folds*, J. Amer. Math. Soc. **1** (1988), no. 1, 117–253.

[38] S. Mukai, *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves*, Nagoya Math. J. **81** (1981), 153–175.

[39] ———, *On the moduli space of bundles on $K3$ surfaces. I*, Vector bundles on algebraic varieties (Bombay, 1984) (Bombay), Tata Inst. Fund. Res. Stud. Math., vol. 11, Tata Inst. Fund. Res., Bombay, 1987, pp. 341–413.

[40] D. O. Orlov, *Equivalences of derived categories and $K3$ surfaces*, J. Math. Sci. (New York) **84** (1997), no. 5, 1361–1381, Algebraic geometry, 7.

[41] ———, *Derived categories of coherent sheaves on abelian varieties and equivalences between them*, Izv. Ross. Akad. Nauk Ser. Mat. **66** (2002), no. 3, 131–158.

[42] ———, *Derived categories of coherent sheaves and equivalences between them*, Uspekhi Mat. Nauk **58** (2003), no. 3, 89–172.

[43] M. Reid, *What is a flip?*, colloquium talk at Utah 1992.

[44] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. (2) **39** (1989), no. 3, 436–456.

[45] ———, *Derived equivalences as derived functors*, J. London Math. Soc. (2) **43** (1991), no. 1, 37–48.

[46] C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics, vol. 1099, Springer-Verlag, Berlin, 1984.

[47] S.-S. Roan, *Minimal resolutions of Gorenstein orbifolds in dimension three*, Topology **35** (1996), no. 2, 489–508.

[48] O. Schiffmann, *Noncommutative projective curves and quantum loop algebras.*, math.QA 0205267.

[49] P. Seidel, *Homological mirror symmetry for the quartic surface*, math.SG/0310414.

[50] ———, *Lectures on four-dimensional Dehn twists*, math.SG/0309012.

[51] P. Seidel and R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), no. 1, 37–108.

[52] M. Thaddeus, *Geometric invariant theory and flips*, J. Amer. Math. Soc. **9** (1996), no. 3, 691–723.

[53] K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*, Springer-Verlag, Berlin, 1975, Notes written in collaboration with P. Cherenack, Lecture Notes in Mathematics, Vol. 439.

[54] M. Van den Bergh, *Non-commutative crepant resolutions*, math.RA/0211064, to appear in the Proceedings of the Abel Bicentennial Conference.

[55] ———, *Three-dimensional flops and non-commutative rings*, math.AG/0207170, to appear in Duke Math. Journal.

[56] J.-L. Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253 pp. (1997), With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.

Lutz Hille
Mathematisches Seminar
Universität Hamburg
D-20 146 Hamburg
Germany
E-mail: hille@math.uni-hamburg.de
http://www.math.uni-hamburg.de/home/hille/

Michel Van den Bergh
Departement WNI
Limburgs Universitair Centrum
Universitaire Campus
3590 Diepenbeek
Belgium
E-mail: vdbergh@luc.ac.be
http://alpha.luc.ac.be/Research/Algebra/Members/michel_id.html