STOCHASTIC STOPPING IN RANDOM WALKS

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1. Description

Let X be a random walk where the steps S follow a distribution θ. Assume we start in $-u$ for $u \geq 0$ and we stop when $X \geq 0$. In that case the value of X is called the *overshoot*. In many cases the overshoot is undesirable. For example in the Sequential Probability Ratio Test (SPRT) it affects the characteristics. Below we sketch a simple procedure which makes the random walk stop exactly at 0 on average (conditioned on the random walk stopping at all, which is not automatic if $E(S) < 0$).

We will do this via function $q(\delta)$ such that if we are in position δ then we will either stop with probability $1 - q(\delta)$ or continue with probability $q(\delta)$. The requirement that on average the random walk should stop exactly at zero leads to the following condition for $q := q(\delta)$.

$$0 = (1-q)(-\delta) + q\int_{\delta}^{\infty}(y-\delta)\theta(y)dy$$

which solves to

$$q = \frac{\delta}{\delta + \int_{\delta}^{\infty}(y-\delta)\theta(y)dy} \tag{1}$$

If the distribution θ is discrete then the integral in the denominator becomes of course a sum.

The quantity $\int_{\delta}^{\infty}(y-\delta)\theta(y)dy$ is interesting. It turns out that in practice one can approximate it by replacing θ by a normal distribution $\phi((x-\mu)/\sigma)/\sigma$ with the same variance σ^2 and expectation value μ. In that case we compute

$$\begin{aligned}\int_{\delta}^{\infty}(y-\delta)\theta(y)dy &\cong \int_{\delta}^{\infty}\frac{y-\delta}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right)dy \\ &= \int_{\sigma z+\mu\geq\delta}(\sigma z+\mu-\delta)\phi(z)dz \\ &= \sigma\int_{\delta_z}^{\infty}(z-\delta_z)\phi(z)dz\end{aligned}$$

with $\delta_z = (\delta - \mu)/\sigma$. The standard normal distribution has the interesting property

$$\phi'(z) = -z\phi(z)$$

so that we obtain

$$\int_{\delta_z}^{\infty}(z-\delta_z)\phi(z)dz = \phi(\delta_z) - \delta_z(1-\Phi(\delta_z))$$

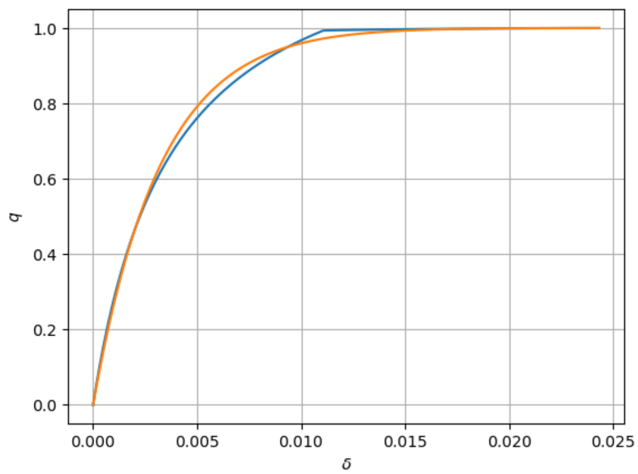

FIGURE 1.

for Φ the cumulative density function of ϕ. So substituting in (1) we get the final formula

$$q \cong \frac{\delta}{\delta + \sigma(\phi(\delta_z) - \delta_z(1 - \Phi(\delta_z)))}$$

2. EXAMPLE

Consider the following distribution

probability	value
0.00625	-0.02251619999999998600
0.23784	-0.01132079999999999000
0.50951	-0.00012539999999999427
0.23983	0.01107000000000000300
0.00657	0.02226539999999999800

Figure 1 shows both the exact value of q and the approximated one. Figure 2 shows both graphs if we are only allowed to stop every two steps.

3. PRACTICAL IMPLEMENTATION

The algorithm was implemented in the SPRT simulator `https://github.com/vdbergh/simul` and performs entirely satisfactorily.

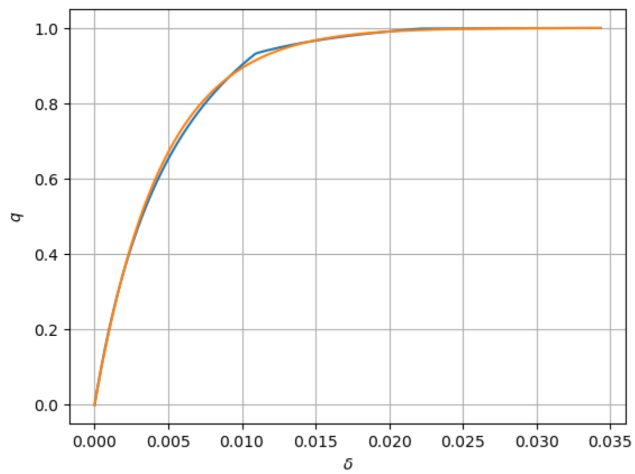

Figure 2.