THE SPRT FOR A BROWNIAN MOTION

1. Main formulas

1.1. The general case. For $a < 0 < b$ define the functions

$$\begin{aligned} P(h) &= \frac{1 - e^{-ha}}{e^{-hb} - e^{-ha}}, \\ T(h) &= \frac{2}{h}(bP(h) + a(1 - P(h))) \\ &= \frac{2}{h} \frac{ae^{-hb} - be^{-ha} + (b - a)}{e^{-hb} - e^{-ha}}. \end{aligned}$$

To evaluate P, T for $h = 0$ we may use L'Hôpital's rule. We find

$$\begin{aligned} P(0) &= -\frac{a}{b - a}, \\ T(0) &= -ab. \end{aligned}$$

Theorem 1.1. *Consider a Brownian motion with drift μ and infinitesimal variance σ^2. Assume we perform an SPRT test for $H0 : \mu = \mu_0$ versus $H1 : \mu = \mu_1$ with error probalities α, β. Define*

$$\begin{aligned} h_\mu &= \frac{2\mu - (\mu_0 + \mu_1)}{\mu_1 - \mu_0}, \\ w &= \frac{\mu_1 - \mu_0}{\sigma}. \end{aligned}$$

Then the pass probability p and the expected duration T of the test are respectively given by

$$\begin{aligned} p &= P(h_\mu), \\ T &= \frac{T(h_\mu)}{w^2}, \end{aligned}$$

where as usual

$$\begin{aligned} a &= \log\left(\frac{\beta}{1 - \alpha}\right), \\ b &= \log\left(\frac{1 - \beta}{\alpha}\right). \end{aligned}$$

Remark 1.2. Note that h_μ represents simply the *relative position* of μ with respect to the interval $[\mu_0, \mu_1]$ with $h_\mu = -1$ corresponding to $\mu = \mu_0$ and $h_\mu = 1$ corresponding to $\mu = \mu_1$.

On the other hand w represents a universal measure for the width of the interval $[\mu_0, \mu_1]$. This may be motivated by the fact that the expected distance covered by the diffusion component of a Brownian motion is $\sqrt{t}\sigma$. We find that, all else being equal, the expected duration of the test is proportional to σ^2.

1.2. **The case of a symmetric test.** Assume now $\alpha = \beta$. In that case

$$b = -a = \log\left(\frac{1-\alpha}{\alpha}\right)$$

E.g. in the iconic case $\alpha = \beta = 0.05$ we find

$$b = -a = \log(19) = 2.94444\cdots$$

In this case the universal functions $P(h)$ and $T(h)$ simplify as follows

$$P(h) = \frac{1}{1+e^{-hb}},$$
$$T(h) = \frac{2b}{h}\frac{1-e^{-hb}}{1+e^{-hb}}.$$

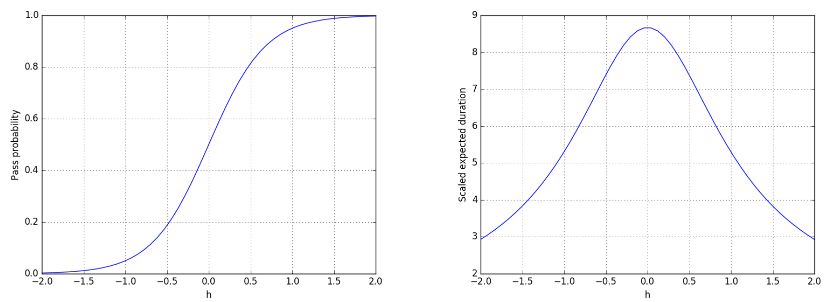

FIGURE 1. Graphs of the universal functions P and T for $\alpha = \beta = 0.05$.

2. Explanation

Consider a Brownian random walk, starting in 0 with bounds $a < 0 < b$, drift $\bar{\mu}$ and infinitesimal variance $\bar{\sigma}^2$. Put

$$h := 2\frac{\bar{\mu}}{\bar{\sigma}^2}.$$

The probability of touching the bound b first is

$$p_b := \frac{1-e^{-ha}}{e^{-hb}-e^{-ha}}.$$

Hence the probability of touching the bound a first is

$$p_a := 1 - p_b.$$

By the optional stopping theorem we find that the expected time before the random walk touches one of the bounds is

$$T := \frac{p_a a + p_b b}{\bar{\mu}} = \frac{1}{\bar{\mu}}\frac{ae^{-hb} - be^{-ha} + (b-a)}{e^{-hb}-e^{-ha}}.$$

Now assume that we have a Browian motion with drift μ and infinitesimal variance σ^2 and we perform an SPRT for $H0 : \mu = \mu_0$ versus $H1 : \mu = \mu_1$. If the Brownian motion is at position s at time t then one computes that the log likelihood ratio is

$$\text{LLR} = \frac{(\mu_1 - \mu_0)(2s - t(\mu_0 + \mu_1))}{2\sigma^2}.$$

From this one computes that LLR itself follows a Brownian motion with drift and infinitesimal variance given by

$$\bar{\mu} = \frac{(\mu_1 - \mu_0)(2\mu - (\mu_0 + \mu_1))}{2\sigma^2},$$

$$\bar{\sigma}^2 = \left(\frac{\mu_1 - \mu_1}{\sigma}\right)^2.$$

We obtain

$$h = \frac{2\mu - (\mu_0 + \mu_1)}{\mu_1 - \mu_0},$$

$$\bar{\mu} = \frac{h}{2}\left(\frac{\mu_1 - \mu_0}{\sigma}\right)^2.$$

It now suffices to substitute this into the formulas for p_b and T.